DIE NATÜRLICHEN GLEICHUNGEN DER ANALYTISCHEN CURVEN IM EUKLIDISCHEN RAUME*

VON

E. STUDY

Die Grundformeln der gewöhnlichen Differentialgeometrie (und nicht minder zum Beispiel auch die der Differentialgeometrie im Nicht-Euklidischen Raume) haben sämmtlich ihre Quelle in einigen wenigen algebraischen Identitäten, Determinantenrelationen, *die sich im Voraus erschöpfend angeben lassen.* Indem man diesen Gedanken, der in älteren Darstellungen wohl noch nie zu seinem Rechte gekommen ist, sich deutlich vor Augen hält, kommt man zu einer neuen Behandlungsweise des Stoffes, die der Verfasser, zunächst allerdings nur an dem Beispiele der Curven, vor kurzem auseinandergesetzt hat. †

Die gegenwärtige Arbeit enthält eine erste Fortsetzung der genannten Untersuchung. Für den besonders wichtigen Fall, dass der Bogen (oder, bei den Minimalcurven der natürliche Parameter, d. h. einer dieser Parameter) die unabhängige Veränderliche ist, werden vollständige Systeme von Differentialinvarianten aufgestellt (§ 2 und § 5). Als Anwendung dieser kleinen ganz elementaren Theorie ergiebt sich die Thatsache, dass das Integrationsproblem der natürlichen Gleichungen einer regulären Curve äquivalent ist mit dem Problem der Integration gewisser linearer Differentialgleichungen, deren Theorie demgemäss vollständig und mit einfachen Hülfsmitteln entwickelt wird (§6–§8). Im Besonderen folgt hieraus, dass man die natürlichen Gleichungen aller der regulären Curven kennzeichnen und durch Quadraturen auflösen kann, deren Tangenten parallel sind zu den Erzeugenden irgend eines Kegels zweiter Ordnung. Die bekannte Theorie der durch die Gleichung $R : T = \text{const.}$ charakterisierten Schraubenlinien (die Rotationskegeln entsprechen) ist hiervon ein besonderer Fall.

Ausserdem enthält die Arbeit eine Integrationsmethode für die natürlichen Gleichungen selbst (in § 1 und § 3, die für sich ein abgeschlossenes Ganzes bilden.) Diese unsre Methode ist von der sonst üblichen völlig verschieden, wenn auch, wie natürlich, der Charakter der integrierenden Hülfsgleichungen derselbe bleibt. Sie besteht darin, dass der allgemeine Fall zurückgeführt

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† *Zur Differentialgeometrie der analytischen Curven*—weiterhin mit A. C. citiert—Transactions of the American Mathematical Society, vol. 10 (1909).

wird auf das entsprechende Problem für Curven auf gegebenem Minimalkegel. Diese Aufgabe wiederum verlangt die Integration einer *Schwarz*'schen Differentialgleichung (oder einer *Riccati*'schen Gleichung). Gelegentlich ergiebt sich hierbei eine geometrische Deutung des *Schwarz*'schen Differentialausdrucks (§ 1.)

Wegen der ebenen singulären Curven, die hier ausgeschlossen bleiben, verweisen wir auf die citierte Abhandlung (A. C., § 5).

§ 1.

Die Curven auf gegebenem Minimalkegel und die Minimalcurven.

Wir werden zunächst krumme Linien betrachten, die auf dem Minimalkegel

$$y_1^2 + y_2^2 + y_3^2 = 0$$

liegen. Um zu einer zweckdienlichen Parameterdarstellung dieses Kegels zu gelangen, fassen wir ihn als Grenzfall einer Kugel

$$y_1^2 + y_2^2 + y_3^2 = \Re^2$$

von nicht verschwindendem Radius auf. Die letzte Gleichung nun, die wir kürzer auch so schreiben können: $(y|y) = \Re^2$, kann nach bekanntem (unsres Wissens in allem Wesentlichen von Darboux herrührendem) Verfahren, identisch befriedigt werden durch die Substitutionen

$$y_1 \equiv \Re\,\frac{1-st}{s-t}, \qquad y_2 \equiv \Re\cdot i\,\frac{1+st}{s-t}, \qquad y_3 \equiv -\Re\cdot\frac{s+t}{s-t},$$

wobei die uneigentlichen Punkte der Kugel und ausserdem die eigentlichen Punkte von zwei Erzeugenden $\{y_1 - iy_2 = 0,\ y_3^2 - 1 = 0\}$ zunächst ausgeschlossen bleiben. In den Parametern s, t werden dann die automorphen Bewegungen der Kugel, nämlich die Drehungen um deren Mittelpunkt, dargestellt durch irgend zwei mit einander identische lineare Transformationen. Diese lassen sich in der Form

$$s^* = \frac{(\alpha_0 - i\alpha_3)s - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)}, \qquad t^* = \frac{(\alpha_0 - i\alpha_3)t - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)t + (\alpha_0 + i\alpha_3)}$$

schreiben. Die Verhältnissgrössen $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$, deren Quadratsumme nicht verschwinden darf, sind die Euler'schen Parameter der ausgeführten Drehung. Ist diese insbesondere reell, so dürfen auch die Parameter α als reell angenommen werden, und umgekehrt entsprechen reelle Parameter einer reellen Drehung.

Wir führen jetzt einen Grenzübergang aus, indem wir den Quotienten $(s-t):\Re$ durch einen neuen Parameter $2t_1$ ersetzen, und dann zur Grenze

$t = s$, $\Re = 0$ übergehen. So entstehen, wenn für t_1 wieder t geschrieben wird, die Formeln

$$(1) \qquad y_1 \equiv \frac{1 - s^2}{2t}, \qquad y_2 \equiv i\frac{1 + s^2}{2t}, \qquad y_3 \equiv -\frac{2s}{2t}.$$

Diese Gleichungen dienen also zur Parameterdarstellung einer Kugel vom Radius Null, nämlich des Minimalkegels

$$(2) \qquad y_1^2 + y_2^2 + y_3^2 = 0,$$

wobei jedoch der Scheitel dieses Kegels ($t = \infty$) und überhaupt alle Punkte einer Erzeugenden $\{ y_1 - iy_2 = 0,\ y_3 = 0 \}$ in die Parameterdarstellung nicht (nicht unmittelbar) einbezogen sind.

Aus (1) folgt noch

$$(3) \qquad s \equiv \frac{y_1 + iy_2}{y_3} \equiv -\frac{y_3}{y_1 - iy_2}, \qquad t \equiv \frac{1}{y_1 - iy_2} \equiv -\frac{y_1 + iy_2}{y_3^2}.$$

Für die Parameter s und t erhalten wir jetzt — durch den gleichen Grenzübergang — die Transformationsgleichungen

$$(4) \quad s^* = \frac{(\alpha_0 - i\alpha_3)s - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)}, \qquad t^* = \frac{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}{\{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)\}^2} \cdot t,$$

die zur Darstellung der Drehungen um den Scheitel des Kegels dienen, soweit geeignete Punkte des Kegels selbst in Frage kommen. Die Formeln (4) aber setzen folgenden Lehrsatz in Evidenz:

Bei geeigneter Coordinatenwahl vertauscht die Gruppe der automorphen Bewegungen eines Minimalkegels dessen Punkte durch die erste Lie'sche Erweiterung der Gruppe aller linearen Transformationen eines binären Gebietes.

Das heisst, denkt man sich s als Function eines Parameters p, so zeigt die Transformationsgleichung für t, wie der Differentialquotient $ds : dp$ transformiert wird. Der Satz ist hier deshalb von Bedeutung, weil er die Motivierung unseres Grenzüberganges vervollständigt. Er zeigt, warum es unter Umständen unzweckmässig sein kann, an Stelle des Zeichens t ein Zeichen für $1 : t$ zu gebrauchen, was bei anderer Art der Exposition näher liegen würde. Insbesondere führt unsere Bezeichnung noch zu einer bemerkenswerthen Zusammenfassung der beiden Gleichungen (4) in eine einzige ("duale") Gleichung:

$$(5) \qquad s^* + \epsilon t^* = \frac{(\alpha_0 - i\alpha_3)(s + \epsilon t) - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)(s + \epsilon t) + (\alpha_0 + i\alpha_3)} \qquad (\epsilon^2 = 0).^*$$

Nach diesen Vorbereitungen können wir nun jede *krumme* analytische Linie

* Vgl. des Verfassers *Geometrie der Dynamen* (Leipzig, 1903), II, § 23, und A. C., S. 40, 41, wo umfassendere Formeln angegeben werden.

auf dem Minimalkegel, oder doch ein Stück einer solchen Linie, dadurch ausdrücken, dass wir s und t als analytische Functionen eines Parameters p darstellen, als Functionen, die natürlich einen gemeinsamen Existenzbereich haben müssen und von denen übrigens s nicht constant sein darf. Unter allen Parameterdarstellungen einer solchen Curve giebt es aber eine, und von einer unwesentlichen Willkür abgesehen, auch nur eine ausgezeichnete, die durch den *natürlichen Parameter* der Curve. *Um auszudrücken, dass p ein "natürlicher" Parameter sei, genügt nicht eine einzige Differentialgleichung, sondern es sind* — wie A. C., S. 34 gezeigt worden ist — *deren zwei erforderlich,*

$$(y'|y')_p \equiv -1, \qquad (yy'y'')_p \equiv -1,$$

die natürlich mit einander verträglich sein müssen und es thatsächlich auch sind. Man hat nun in unserem Falle, wo y durch Vermittelung von s und t von p abhängt,

$$(y'|y')_p \equiv -\left(i\frac{s'}{t}\right)^2, \qquad (yy'y'')_p \equiv -\left(i\frac{s'}{t}\right)^3,$$

also

$$t \equiv i\frac{ds}{dp}. \tag{6}$$

Die auf den nunmehr natürlichen Parameter p bezogene Curve wird mithin durch folgende Formeln dargestellt:

$$y_1 \equiv -i\frac{1-s^2(p)}{2s'(p)}, \qquad y_2 \equiv \frac{1+s^2(p)}{2s'(p)}, \qquad y_3 \equiv i\frac{2s(p)}{2s'(p)}. \tag{7}$$

Die *natürliche Gleichung* der Curve auf dem Minimalkegel (2) aber giebt das Quadrat der Krümmung als Function von p (A. C., Nr. 39, S. 35):

$$\phi(p) \equiv \frac{1}{R^2} \equiv (y''|y'')_p \equiv -\frac{2s'_p s'''_p - 3s''^2_p}{s'^2_p s'^2_p}. \tag{8}$$

Man hat also zur Bestimmung der Curve aus ihrer natürlichen Gleichung die Schwarz'sche Differentialgleichung (8) *oder die mit dieser äquivalente, durch die Substitution*

$$\frac{s''}{2s'} \equiv \omega$$

entstehende Riccati'sche Gleichung

$$\frac{d\omega}{dp} \equiv -\frac{\phi}{4} + \omega^2 \tag{9}$$

aufzulösen.

Wir erinnern daran, dass durch drei verschiedene Lösungen der Riccati'schen Gleichung eine Lösung der Schwarz'schen Gleichung auf die einfachste Weise

ausgedrückt werden kann:

$$s \equiv \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}.*$$

Zugleich ist bewiesen:

Zu jeder Schwarz'schen Differentialgleichung

$$\{s, p\} \equiv \frac{2s's''' - 3s''s''}{2s's'} \equiv -\tfrac{1}{2}\phi(p) \qquad \text{(Nr. 8)}$$

gehört eine Classe unter einander congruenter krummer Linien, die auf dem Minimalkegel $y_1^2 + y_2^2 + y_3^2 = 0$ *liegen, und zu jeder einzelnen Lösung der Gleichung eine bestimmte unter diesen Curven.* p *(allgemein* $p + const$*) ist ein zugehöriger natürlicher Parameter,* ϕ *die in der gemeinsamen natürlichen Gleichung jener Curven auftretende Function, also* $\phi(p)$ *das Quadrat ihrer Krümmung an den zu dem Werthe* p *gehörigen Stellen.*

Es ist natürlich nicht ausgeschlossen, dass eine und dieselbe Curve mehreren, auch unendlich vielen Lösungen der Schwarz'schen Gleichung entspricht. Es existiert dann eine Gruppe automorpher Bewegungen der Curve. Continuierlich ist diese Gruppe nur im Falle der Kreise, nämlich der parabolischen (singulären) $\{\phi \equiv 0\}$ und regulären Kreise $\{\phi \equiv \text{const.}, \neq 0\}$.

Zu beachten ist, dass in obigem Satze der natürliche Parameter *nicht* durch den (zu *zweiwerthigem* Differential gehörigen) Curvenbogen ersetzt werden kann. Giebt man den Werth ϕ als Function des Bogens, so ist damit die zugehörige Curve nur bis auf Bewegungen und Umlegungen bestimmt.

Die Gleichung

$$\frac{1}{R^2} \equiv \phi(p)$$

fungiert auch als natürliche Gleichung für die Minimalcurven, die die Curve auf dem Minimalkegel zum sphärischen Bilde, oder zum Bestandtheil ihres sphärischen Bildes haben. (Siehe darüber A. C., § 6, insbes. S. 37.)

Um also eine (*krumme*) Minimallinie aus ihrer natürlichen Gleichung zu bestimmen, hat man nach Integration der Gleichung (8), die mit der Gleichung {A. C., (51)} identisch ist, noch die in der Formel $\int x\,dp$ zusammengefassten drei Quadraturen auszuführen.

§ 2.

Erläuterungen. Differentialinvarianten der Minimalcurven.

Durch das Vorgetragene wird die Frage nach der Bestimmung einer krummen Minimallinie aus ihrer natürlichen Gleichung auf die wohl einfachste Weise

* Der Zusammenhang beider Gleichungen wird eingehend untersucht in der Abhandlung *Über Riccati'sche und Schwarz'sche Differentialgleichungen*, Archiv der Mathematik und Physik, 1910, die weiterhin mit R. Gl. citirt wird.

beantwortet. Man kann sich indessen die Frage vorlegen, wie diese Methode mit mehreren anderen zusammenhängt, die sich ebenfalls darbieten.

Wir beginnen mit der Aufstellung eines in gewissem Sinne vollständigen Systems von Differentialinvarianten einer krummen Minimallinie. Ein solches wird gebildet von der Function ϕ mit ihren Differentialquotienten nach p:

Benutzt man zur Darstellung einer krummen Minimallinie $x \equiv x(p)$ einen natürlichen Parameter, so werden alle auf diesen bezogenen Semiinvarianten $(i\,|\,k)$ und (ikl) ganze rationale Functionen der Function ϕ und ihrer Differentialquotienten, und zwar kommen in diesen Ausdrücken die Differentialquotienten von ϕ höchstens in der $(n-3)$ten Ordnung vor, wenn $i, k, l \leqq n$ sind.

Mit Coefficienten von gleicher Beschaffenheit lassen sich die Semicovarianten $(x^{(k)}\,|\,\omega)$ und $(x^{(i)}x^{(k)}\omega)$ linear ausdrücken durch $(x'\,|\,\omega)$, $(x''\,|\,\omega)$, $(x'''\,|\,\omega)$.

Wir gehen zunächst so weit, wie es die Anwendungen in der Regel erfordern werden, nämlich bis zu dem Werthe $n = 4$:

$$
(1)\qquad
\begin{array}{llll}
(1\,|\,1) \equiv 0, & * , & * , & * , \\
(1\,|\,2) \equiv 0, & (2\,|\,2) \equiv -1, & * , & * , \\
(1\,|\,3) \equiv 1, & (2\,|\,3) \equiv 0 , & (3\,|\,3) \equiv \phi , & * , \\
(1\,|\,4) \equiv 0, & (2\,|\,4) \equiv -\phi, & (3\,|\,4) \equiv \tfrac{1}{2}\phi', & (4\,|\,4) \equiv -\phi^2;
\end{array}
$$

$$
\begin{array}{ll}
(123) \equiv -1, & (124) \equiv 0 , \\
(134) \equiv \phi , & (234) \equiv -\tfrac{1}{2}\phi',*
\end{array}
$$

Man sieht, dass, wenn der Satz für den Werth $n-1\,(\geqq 4)$ als richtig angenommen wird, er auch für den Werth n gelten muss. In der That erhält man aus den Semiinvarianten $(i\,|\,k)$ für $i, k \leqq n-1$ die Semiinvarianten $(k\,|\,n)$ für $k < n$ durch je einmalige Differentiation. Durch diese Grössen lassen sich sodann die Semiinvarianten $(ikn)\,\{i < k < n\}$ ganz und rational ausdrücken, wie man durch Bildung des Productes $-(123)(ikn)$ erkennt. Durch Entwickelung des Productes $-(123)(n\,|\,n)$ erhält man sodann den Werth der noch fehlenden Semiinvariante $(n\,|\,n)$. Entsprechend beweist man den Satz über die Semicovarianten, von denen wir nur einige wenige anführen wollen:

$$
(2)\quad (12\omega) \equiv -(1\,|\,\omega), \quad (13\omega) \equiv -(2\,|\,\omega), \quad (23\omega) \equiv \phi\cdot(1\,|\,\omega) - (3\,|\,\omega);
$$

$$
(3)\qquad (4\,|\,\omega) \equiv \tfrac{1}{2}\phi'\cdot(1\,|\,\omega) + \phi\cdot(2\,|\,\omega).
$$

Die wichtigsten unter diesen Gleichungen sind die erste unter (2) und die Gleichung (3). Diese letzte ist offenbar mit der natürlichen Gleichung der

* Diese Tafel ergiebt sich aus den Formeln A. C., Nr. 36, 37, wenn man linker Hand jeden Differentiationsindex durch den nächsthöheren ersetzt.

Minimalcurve äquivalent, da man ihr die Function $\phi(p)$ entnehmen kann. Sie zeigt, dass die Coordinaten $y_k \equiv dx_k/dp$ des Differentialvectors $y \equiv x_p'$ Lösungen einer linearen Differentialgleichung 3. Ordnung sind, die, wegen der ersten Gleichung unter (2), mit ihrer Adjungierten zusammenfällt. *Diese Differentialgleichung*

$$y''' - \phi(p)\cdot y' - \tfrac{1}{2}\phi'(p)\cdot y = 0 \tag{4}$$

ist aber (bekanntlich) die allgemeinste vom zweiten Gliede befreite lineare Gleichung 3. Ordnung, bei der zwischen je drei linear-unabhängigen Lösungen eine homogene quadratische Gleichung besteht. Man kann daher die Theorie der krummen Linien auf Minimalkegeln und weiterhin die Theorie der krummen Minimallinien auch als eine Anwendung der Theorie dieser speciellen linearen Differentialgleichungen 3. Ordnung behandeln. Wie dieser Gedanke weiter auszuführen ist, liegt auf der Hand. Wir verweisen auf die citierte Arbeit (R. Gl., § 12), wo insbesondere auch der Zusammenhang der angedeuteten Methode mit der zuvor angewendeten dargelegt ist.

Als eine weitere Folgerung unserer Theorie heben wir hervor:

Wie zu den regulären Curven, so gehört auch zu den Minimalcurven im Allgemeinen ein System von orthogonalen Triedern, ein die Curve begleitendes Axenkreuz, und entsprechend ein System von Formeln, die den Serret-Frenet'schen Gleichungen analog sind.

Der hier zu Grunde liegende Gedanke ist ganz selbstverständlich, es scheint uns aber nöthig, die correkte Form eines derartigen Satzes festzustellen.

Das angedeutete System von drei zu einander senkrechten Axen (und Einheitsvectoren) existiert an allen den Stellen regulären Verhaltens einer Minimalcurve, wo das Krümmungsmaass ihres sphärischen Bildes nicht verschwindet ($\phi \neq 0$). *Es existiert also überhaupt nicht im Falle der Minimalcurven 3. Ordnung* (*wo* $\phi \equiv 0$ *ist*).

Nennen wir $1/R$ und $1/T$ die Krümmung und die Torsion des sphärischen Bildes unserer Curve, so haben wir, wie bereits abgeleitet (A. C., S. 34),

$$\frac{1}{R} \equiv \sqrt{\phi}, \qquad \frac{1}{T} \equiv \tfrac{1}{2}\frac{\phi'}{\phi} \equiv -\frac{1}{R}\frac{dR}{dp}. \tag{5}$$

Dem Bogen des sphärischen Bildes kann irgend einer der in der Formel

$$s \equiv \sqrt{-1}\, p \;\{+\text{const.}\} \tag{6}$$

zusammengefassten Werthe beigelegt werden. Für die Einheitsvectoren α, β, γ, die zu Tangente, Hauptnormale und Binormale des sphärischen Bildes gehören, —derart, dass

$$(\alpha\beta\gamma) \equiv 1, \quad (\beta\gamma\omega) \equiv (\alpha|\omega), \quad (\gamma\alpha\omega) \equiv (\beta|\omega), \quad (\alpha\beta\omega) \equiv (\gamma|\omega)$$

wird — ergeben sich sodann die Ausdrücke

$$(7)\qquad (\alpha|\omega)\equiv -\sqrt{-1}(x''|\omega),\qquad (\beta|\omega)\equiv -R\cdot(x'''|\omega),$$
$$(\gamma|\omega)\equiv \sqrt{-1}\cdot R\cdot(x''x'''\omega),$$

wobei die Differentiationen sich, wie zuvor, auf den natürlichen Parameter p beziehen. An den Stellen, wo die Vectoren α, β, γ existieren, kann man dann alle Differentialvectoren x', x'', x''', $\cdots$, linear durch sie ausdrücken, zum Beispiel:

$$(8)\qquad x'\equiv -R\{\beta+\sqrt{-1}\gamma\},\qquad x''\equiv\sqrt{-1}\cdot\alpha,\qquad x'''\equiv -\frac{1}{R}\cdot\beta,$$
$$x''''\equiv\frac{1}{R^2}\cdot\left\{\sqrt{-1}\cdot\alpha+\frac{dR}{dp}(\beta+\sqrt{-1}\gamma)\right\}.$$

Die *Frenet*'schen Gleichungen für das sphärische Bild nehmen jetzt die Form an:

$$(9)\qquad \frac{d\alpha}{dp}\equiv\frac{\sqrt{-1}}{R}\cdot\beta,\quad \frac{d\beta}{dp}\equiv-\frac{\sqrt{-1}}{R}\left\{\alpha+\frac{dR}{dp}\gamma\right\},\quad \frac{d\gamma}{dp}\equiv\frac{\sqrt{-1}}{R}\cdot\frac{dR}{dp}\cdot\beta.$$

Man sieht, dass alle diese Formeln dort illusorisch werden, wo die Function ϕ verschwindet, nicht aber an anderen Stellen regulären Verhaltens von ϕ, oder — was dasselbe ist — nicht an anderen Stellen regulären Verhaltens der Curve. Man mag diese letzten Stellen ($\phi\neq 0$) etwa solche *gewöhnlichen Verhaltens* nennen. Freilich bestehen dann die Minimalcurven 3. Ordnung aus lauter Stellen nicht "gewöhnlichen" Verhaltens.*

Die Gleichungen (9) können schliesslich nach der bei regulären Curven üblichen Methode behandelt werden. Damit hätten wir einen dritten Ansatz zur Bestimmung der durch die Ungleichung $\phi(p)\not\equiv 0$ charakterisierten Minimalcurven aus einer natürlichen Gleichung. Die angedeutete Methode besteht darin, dass man für irgend drei gleichnamige Coordinaten α_κ, β_κ, γ_κ der Einheitsvectoren α, β, γ der Reihe nach Ausdrücke der Form

$$\frac{1-\sigma_\kappa\tau_\kappa}{\sigma_\kappa-\tau_\kappa},\quad i\frac{1+\sigma_\kappa\tau_\kappa}{\sigma_\kappa-\tau_\kappa},\quad -\frac{\sigma_\kappa+\tau_\kappa}{\sigma_\kappa-\tau_\kappa}$$

setzt. Die Grössen σ_κ, τ_κ genügen dann für $\kappa=1,2,3$ sämmtlich der Riccati'schen Gleichung

$$(10)\qquad \frac{d\Omega}{dp}\equiv\frac{1}{2R}\left\{\frac{dR}{dp}+2\Omega-\frac{dR}{dp}\Omega^2\right\},$$

deren vollständige Lösung, wie bekannt, die der Gleichungen (9) nach sich zieht, und damit, nach (8), auch $y\equiv x'_p$ und dann durch drei Quadraturen x liefert.

* Man kann übrigens bemerken, dass an Stelle der Gleichungen (7) noch etwas einfachere treten, wenn man anstatt p als Parameter das Integral über den Contingenzwinkel $\sqrt{-1}\,dp:R$ des sphärischen Bildes einführt.

Aber die skizzierte Methode steht hinter unserer ersten zurück, und auch bei Erhaltung ihres Grundgedankens lässt sie sich durch eine einfachere ersetzen.

Der Vortheil, den die Einführung des beweglichen orthogonalen Trieders bietet, beruht zum Theil darauf, dass alle solchen Trieder zu einander congruent sind. Der gleichen Eigenschaft erfreuen sich aber auch Trieder, und zugehörige Tripel gewisser Vectoren a, b, c, die den folgenden Gleichungen genügen:

$$\begin{aligned} &(a|a) = 0, \quad (b|b) = -1, \quad (c|c) = 0, \\ &(b|c) = 0, \quad (c|a) = -1, \quad (a|b) = 0, \\ &(abc) = 1, \end{aligned} \tag{11}$$

und für die dann weiter auch die Gleichungen gelten:

$$(bc\omega) = -(c|\omega), \quad (ca\omega) = -(b|\omega), \quad (ab\omega) = -(a|\omega). \tag{12}$$

Ein derartiges Tripel von Vectoren, deren zwei die Länge Null haben, setzen wir jetzt an Stelle der drei Einheitsvectoren α, β, γ, nämlich dieses:

$$\begin{aligned} &a \equiv y \equiv x_p', \qquad b \equiv y' \equiv x_p'', \\ &c \equiv \tfrac{1}{2}\phi \cdot y - y'' \equiv \tfrac{1}{2}\phi \cdot x' - x'''. \end{aligned} \tag{13}$$

Die Formeln zeigen, dass dieses neue System linear-unabhängiger Vectoren, abweichend von dem vorigen, an allen regulären Stellen der betrachteten Minimalcurve existiert, und dass also auch die Minimalcurven 3. Ordnung und ihre sphärischen Bilder hier nicht mehr ausgeschlossen zu werden brauchen. An Stelle der Frenet'schen Gleichungen (9) treten jetzt andere, die insofern einen noch etwas einfacheren Bau haben, als in ihnen die zur Bestimmung von R dienende Quadratwurzel und der Differentialquotient der Function ϕ nicht vorkommen:

$$\frac{da}{dp} \equiv b, \quad \frac{db}{dp} \equiv \tfrac{1}{2}\phi \cdot a - c, \quad \frac{dc}{dp} \equiv -\tfrac{1}{2}\phi \cdot b. \tag{14}$$

Bilden wir nun das Product $(abc)\cdot(\omega|\omega)$, so erhalten wir

$$(\omega|\omega) \equiv -2(a|\omega)(c|\omega) - (b|\omega)^2;$$

es genügen also je drei gleichnamige Coordinaten a_κ, b_κ, c_κ der Vectoren a, b, c der Gleichung

$$2a_\kappa c_\kappa + b_\kappa^2 = -1 \qquad (\kappa = 1, 2, 3).$$

Diese Gleichung wird identisch befriedigt, wenn wir für die drei genannten Coordinaten der Reihe nach Ausdrücke der Form

$$-i\frac{1}{\mu_\kappa - \nu_\kappa}, \qquad i\frac{\mu_\kappa + \nu_\kappa}{\mu_\kappa - \nu_\kappa}, \qquad i\frac{2\mu_\kappa\nu_\kappa}{\mu_\kappa - \nu_\kappa}$$

einsetzen. Die Gleichungen (14) gehen dann über in die Gleichungen

$$\frac{d\mu}{dp} \equiv -\frac{\phi}{4} + \mu^2, \qquad \frac{dv}{dp} \equiv -\frac{\phi}{4} + v^2; \tag{15}$$

μ und v sind also in jedem der drei Fälle ($\kappa = 1, 2, 3$) Lösungen eben der Riccati'schen Gleichung (§ 1, Nr. 9), *die wir zuvor auf anderem Wege gewonnen hatten.* Man kann daher, wie zuvor, nach vollständiger Lösung der Gleichung (9) in § 1 die Coordinaten des gesuchten Vectors $y \equiv x'$ unmittelbar hinschreiben. —

Aus unserer Darlegung geht hervor, dass in dem Bereiche ($\phi \neq 0$), in dem die Gleichungen (§ 1, Nr. 9) *und* (§ 2, Nr. 10) *zugleich Geltung haben, sie mit einander äquivalent sein müssen.* In der That lassen sie sich auch in einander transformieren.

Zunächst wird der Zusammenhang zwischen den Vectorentripeln a, b, c und α, β, γ gefunden:

$$\alpha \equiv -\sqrt{-1}\, b, \qquad \beta \equiv -R\left\{\frac{a}{2R^2} - c\right\}, \qquad \gamma \equiv \sqrt{-1}\, R\left\{\frac{a}{2R^2} + c\right\}. \tag{16}$$

Setzen wir zum Beispiel $\sqrt{-1} = i$, so ergiebt sich der Zusammenhang zwischen den mit σ, τ und μ, ν bezeichneten Grössen in jedem Falle ($\kappa = 1, 2, 3$) in der Form

$$\sigma_\kappa \equiv \frac{2R\mu_\kappa - 1}{2R\mu_\kappa + 1}, \qquad \tau_\kappa \equiv \frac{2Rv_\kappa - 1}{2Rv_\kappa + 1}. \tag{17}$$

Daher wird jeder Lösung ω der Gleichung

$$\frac{d\omega}{dp} \equiv -\frac{\phi}{4} + \omega^2 \qquad \left\{\phi \equiv \frac{1}{R^2}\right\}$$

eine Lösung Ω der Gleichung

$$\frac{d\Omega}{dp} \equiv \frac{1}{2R}\left\{\frac{dR}{dp} + 2\Omega - \frac{dR}{dp}\Omega^2\right\}$$

zugeordnet, und umgekehrt, durch die Substitutionen

$$\Omega \equiv \frac{2R\omega - 1}{2R\omega + 1}, \qquad \omega \equiv \frac{1}{2R}\frac{1 + \Omega}{1 - \Omega}. \tag{18}$$

Aber dieser leicht zu verificierende Satz hat zur selbstverständlichen Voraussetzung, dass ϕ nicht verschwindet. Die durch ω und Ω vermittelten Lösungen des Problems der natürlichen Gleichungen dürfen daher, wenn man genau sein will, *nicht* schlechthin "äquivalent" genannt werden.

§ 3. *Die regulären Curven.*

Das Problem der Bestimmung einer regulären Curve aus ihren natürlichen Gleichungen

$$\Phi \equiv \Phi(s), \qquad \Psi \equiv \Psi(s) \qquad (\Phi \not\equiv 0) \tag{1}$$

lässt sich auf die in § 1 behandelte Aufgabe zurückführen.

Wie wir wissen, gehören zu jedem Punkte x allgemeiner Lage (A. C., S. 20) der Curve (x) zwei Punkte, von denen aus der zugehörige Krümmungskreis durch Minimalkegel projiciert wird, die beiden *Scheitel* des Punktes x (A. C., S. 47, 48). Nennen wir diese Punkte $x + \xi$ und $x + \eta$, und denken wir uns die Vectoren ξ, η mit ihren Anfangspunkten am Anfangspunkte der Coordinaten angeheftet, so erhalten wir zwei neue auf dem zugehörigen Minimalkegel gelegene Curven $\xi \equiv \xi(s)$, $\eta \equiv \eta(s)$. Diese (die in ihrer ganzen Erstreckung betrachtet, nicht nothwendig von einander verschieden sind) wollen wir die beiden *Begleiter* der regulären Curve nennen. Es besteht nun der Satz:

Sind die natürlichen Gleichungen einer regulären Curve bekannt, so lässt sich die natürliche Gleichung eines jeden ihrer Begleiter explicite angeben. Nach Integration einer dieser beiden Gleichungen findet man die Curve selbst durch Quadratur über ein Vectordifferential.

Wir construieren zunächst, unter Benutzung des Bogens als Parameter, das die reguläre Curve begleitende System von drei Einheitsvectoren (A. C., S. 21, Nr. 6):

$$\begin{gathered}(\alpha|\omega) \equiv (x'|\omega) \equiv (y|\omega), \qquad (\beta|\omega) \equiv R(x''|\omega) \equiv R(y'|\omega), \\ (\gamma|\omega) \equiv R(x'x''\omega) \equiv R(yy'\omega).\end{gathered} \tag{2}$$

Dann gelten die *Frenet-Serret'schen Gleichungen*,

$$\frac{d\alpha}{ds} \equiv \frac{\beta}{R}, \qquad \frac{d\beta}{ds} \equiv -\frac{\alpha}{R} + \frac{\gamma}{T}, \qquad \frac{d\gamma}{ds} \equiv -\frac{\beta}{T}. \tag{3}$$

Die beiden Begleiter werden jetzt gegeben durch die Formeln

$$\xi \equiv R(\beta + i\gamma), \qquad \eta \equiv R(\beta - i\gamma). \tag{4}$$

Hieraus folgt, wenn

$$\begin{aligned} M &\equiv \frac{d\lg R}{ds} - \frac{i}{T} \equiv -\frac{1}{\Phi}\left\{\frac{1}{2}\frac{d\Phi}{ds} + i\Psi\right\}, \\ N &\equiv \frac{d\lg R}{ds} + \frac{i}{T} \equiv -\frac{1}{\Phi}\left\{\frac{1}{2}\frac{d\Phi}{ds} - i\Psi\right\}, \end{aligned} \tag{5}$$

gesetzt wird,

$$\frac{d\xi}{ds} \equiv -\alpha + M\xi, \qquad \frac{d\eta}{ds} \equiv -\alpha + N\eta, \tag{6}$$

und

$$\frac{d^2\xi}{ds^2} \equiv -M\cdot\alpha - \frac{1}{R}\cdot\beta + \left\{ M^2 + \frac{dM}{ds} \right\}\cdot\xi,$$

$$\frac{d^2\eta}{ds^2} \equiv -N\cdot\alpha - \frac{1}{R}\cdot\beta + \left\{ N^2 + \frac{dN}{ds} \right\}\cdot\eta;$$

$$(7) \qquad (\xi\xi'\omega) \equiv -i(\xi|\omega), \qquad (\eta\eta'\omega) \equiv i(\eta|\omega);$$

$$(8) \qquad \begin{aligned} (\xi'|\xi') &\equiv 1, \qquad (\xi\xi'\xi'') \equiv i, \\ (\eta'|\eta') &\equiv 1, \qquad (\eta\eta'\eta'') \equiv -i; \end{aligned}$$

also, wenn die natürlichen Parameter der beiden Begleiter p und q gennant werden,

$$\frac{dp}{ds} \equiv \frac{(\xi\xi'\xi'')}{(\xi'|\xi')} \equiv i, \qquad \frac{dq}{ds} \equiv \frac{(\eta\eta'\eta'')}{(\eta'|\eta')} \equiv -i \text{ (A. C., Nr. 35)},$$

$$(9) \qquad p \equiv is \text{ (+ const.)}, \qquad q \equiv -is \text{ (+ const.)}.$$

Nennen wir ferner $\phi(s)$ und $\psi(s)$ die quadrierten Krümmungsmaasse der beiden begleitenden Curven, so folgt

$$(10) \qquad \begin{aligned} \phi(s) &\equiv (\xi''|\xi'')_s \equiv \Phi - 2\frac{dM}{ds} - M^2, \\ \psi(s) &\equiv (\eta''|\eta'')_s \equiv \Phi - 2\frac{dN}{ds} - N^2. \end{aligned}$$

Mithin erhalten wir nach Lösung der beiden Schwarz'schen Gleichungen

$$(11) \qquad \{\sigma, s\} \equiv \tfrac{1}{2}\phi(s), \qquad \{\tau, s\} \equiv \tfrac{1}{2}\psi(s)$$

(Vgl. Nr. 9 und § 1, Nr. 8) die beiden Begleiter:

$$(12) \qquad \begin{aligned} \xi_1 &\equiv \frac{1-\sigma^2}{2\sigma'}, & \eta_1 &\equiv -\frac{1-\tau^2}{2\tau'}, \\ \xi_2 &\equiv i\frac{1+\sigma^2}{2\sigma'}, & \eta_2 &\equiv -i\frac{1+\tau^2}{2\tau'}, \\ \xi_3 &\equiv -\frac{2\sigma}{2\sigma'}, & \eta_3 &\equiv \frac{2\tau}{2\tau'}. \end{aligned}$$

(Die Differentiationen beziehen sich auf s, nicht auf die natürlichen Parameter p, q). Die Formeln Nr. (6) zeigen nunmehr, dass man von den beiden Schwarz'schen Gleichungen (11) nur eine zu lösen nöthig hat, um den Vector α zu finden. Schliesslich erhält man durch Quadratur über α den Punkt x:

$$(13) \qquad -\xi + \int M\xi\, ds \equiv x \equiv -\eta + \int N\eta\, ds.$$

Hiermit ist die Aufgabe vollständig gelöst. Die beiden Vectorintegrale liefern offenbar die zur gesuchten Curve gehörigen *Minimalevoluten*, oder, falls diese nicht existieren, die sie vertretenden Punkte (A. C., S. 48).

Angemerkt zu werden verdient vielleicht noch der Ausdruck

$$\Re = R \cdot T \cdot \sqrt{MN} \tag{14}$$

für den Radius der Schmiegungskugel.

§ 4. *Andere Ableitung des in* § 3 *gefundenen Resultates. Erläuterungen.*

Da die Minimalvectoren ξ, η zu dem Einheitsvector α senkrecht sind, so können die Parameter σ, τ nicht wesentlich verschieden sein von den *symmetrischen Coordinaten des Punktes* α *der Einheitskugel*. Überdies zeigt der in § 1 ausgeführte Grenzübergang, dass beide Arten von Grössen vollkommen identisch sind. Die Umkehrung dieses Gedankenganges muss offenbar zu einer zweiten Herleitung der in § 3 erhaltenen Formeln führen, und auch diese scheint uns Interesse zu bieten.

Wir befriedigen also jetzt die Gleichung

$$y_1^2 + y_2^2 + y_3^2 = 1, \tag{1}$$

in der wir die Coordinaten $y_\varkappa$ mit denen des Vectors α identifizieren ($y \equiv \alpha$) identisch durch die Substitutionen

$$\alpha_1 \equiv \frac{1 - \sigma\tau}{\sigma - \tau}, \qquad \alpha_2 \equiv i\frac{1 + \sigma\tau}{\sigma - \tau}, \qquad \alpha_3 \equiv -\frac{\sigma + \tau}{\sigma - \tau}, \tag{2}$$

wobei nur die Punkte von zwei geradlinigen Erzeugenden der Einheitskugel ausgeschlossen bleiben (§ 1).

Bei Berechnung der Coordinaten des zweiten Einheitsvectors $\beta \equiv R\, d\alpha/ds$ ist zu berücksichtigen, dass sich

$$\left(\frac{ds}{R}\right)^2 \equiv \frac{4 d\sigma\, d\tau}{(\sigma - \tau)^2}$$

ergiebt; es folgt daher

$$\beta_1 \equiv \frac{1}{2R}\left\{\begin{matrix} \frac{1-\sigma^2}{2\sigma'} - \\ -\frac{1-\tau^2}{2\tau'} \end{matrix}\right\}, \quad \beta_2 \equiv \frac{1}{2R}\left\{\begin{matrix} i\frac{1+\sigma^2}{2\sigma'} - \\ -i\frac{1+\tau^2}{2\tau'} \end{matrix}\right\}, \quad \beta_3 \equiv \frac{1}{2R}\left\{\begin{matrix} -\frac{2\sigma}{2\sigma'} + \\ +\frac{2\tau}{2\tau'} \end{matrix}\right\}, \tag{3}$$

$$\gamma_1 \equiv \frac{1}{2R}\left\{\begin{matrix} -i\frac{1-\sigma^2}{2\sigma'} - \\ -i\frac{1-\tau^2}{2\tau'} \end{matrix}\right\}, \gamma_2 \equiv \frac{1}{2R}\left\{\begin{matrix} \frac{1+\sigma^2}{2\sigma'} + \\ +\frac{1+\tau^2}{2\tau'} \end{matrix}\right\}, \quad \gamma_3 \equiv \frac{1}{2R}\left\{\begin{matrix} i\frac{2\sigma}{2\sigma'} + \\ +i\frac{2\tau}{2\tau'} \end{matrix}\right\}. \tag{4}$$

Die Formeln (3) und (4) zeigen, dass die Verbindungen

$$\xi \equiv R(\beta + i\gamma), \qquad \eta \equiv R(\beta - i\gamma) \tag{5}$$

nur von je einer der Grössen σ, τ abhängen: man erhält eben die Formeln (12) des vorigen Paragraphen. Man wird also die Functionen σ und τ einzeln zu bestimmen suchen. Damit kommt man, nach § 1, wieder zu den Formeln (10) und (11) des § 3. Da auch die Gleichungen (6) in § 3 zur Verfügung stehen, so hat man auch auf diese Art die vollständige Lösung des Problems.

Man kann auch so zu Werke gehen, dass man nach Bestimmung etwa des Vectors ξ zunächst den Vector η sucht. Der Vector α, auf den es ankommt, ergiebt sich dann aus der evidenten Gleichung

$$(\xi\eta\omega) \equiv -2iR^2 \cdot (\alpha|\omega), \tag{6}$$

die, im vorigen § angewendet, unmittelbar die Gleichungen (1) geliefert haben würde. Der Vector η aber ergiebt sich, wenn ξ bekannt ist, ohne Weiteres, da offenbar

$$\frac{\xi + \eta}{2} \equiv R^2 \frac{d\alpha}{ds} \equiv \frac{1}{\Phi} \frac{d}{ds} \left\{ M\xi - \frac{d\xi}{ds} \right\} \equiv \frac{1}{\Phi} \frac{d}{ds} \left\{ N\eta - \frac{d\eta}{ds} \right\} \tag{7}$$

ist.

Ferner muss, wenn eine Lösung $\sigma(s)$ oder $\tau(s)$ einer der beiden Schwarz'schen Gleichungen bekannt ist, die entsprechende Lösung $\tau(s)$ oder $\sigma(s)$ der anderen eindeutig sich bestimmen lassen. Man erhält, wenn man in irgend eine der Gleichungen (2) die Werthe der Coordinaten $\alpha_\varkappa$ aus der Gleichung (6) in § 3, und dann die Werthe der Coordinaten $\xi_\varkappa$ oder $\eta_\varkappa$ aus (12) einträgt, unmittelbar die gesuchten Formeln:

$$\tau \equiv \sigma - \frac{2\sigma'\sigma'}{M\sigma' + \sigma''}, \qquad \sigma \equiv \tau - \frac{2\tau'\tau'}{N\tau' + \tau''}. \tag{8}$$

Die Lösungen

$$\mu \equiv \frac{\sigma''}{2\sigma'}, \qquad \nu \equiv \frac{\tau''}{2\tau'} \tag{9}$$

der beiden Riccati'schen Gleichungen

$$\frac{d\mu}{ds} \equiv \frac{\phi}{4} + \mu^2, \qquad \frac{d\nu}{ds} \equiv \frac{\psi}{4} + \nu^2, \tag{10}$$

die den betrachteten Schwarz'schen Gleichungen zugeordnet sind, hängen durch eine lineare Substitution zusammen, die ebenfalls leicht zu berechnen ist. Man hat nämlich, nach (8):

$$M \equiv \frac{2\sigma'}{\sigma - \tau} - \frac{\sigma''}{\sigma'}, \qquad N \equiv \frac{2\tau'}{\tau - \sigma} - \frac{\tau''}{\tau'}. \tag{11}$$

Da ausserdem

$$4\sigma'\tau' \equiv \Phi \cdot (\sigma - \tau)^2 \qquad \text{(Nr. 2)},$$

so folgt

$$\left(M+\frac{\sigma''}{\sigma'}\right)\left(N+\frac{\tau''}{\tau'}\right)+\Phi\equiv 0, \tag{12}$$

oder

$$4\mu\nu+2N\mu+2M\nu+\frac{1}{\Phi^2}\left\{\Phi^3+\Psi^2+\frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2\right\}\equiv 0. \tag{13}$$

Die Discriminante dieser Gleichung ist, wie es sein muss, nicht identisch gleich Null.

Durch das Gesagte ist auch schon die Aufgabe gelöst, die Functionen Φ, Ψ zu finden, wenn die Functionen ϕ, ψ gegeben sind. Nach vollständiger Lösung der beiden Gleichungen (10) oder der entsprechenden Schwarz'schen Gleichungen kennt man σ, τ, M, N, und man hat

$$\Phi\equiv e^{-\int(M+N)ds}\equiv\frac{4\sigma'\tau'}{(\sigma-\tau)^2},\qquad \Psi\equiv\frac{i}{2}\Phi\cdot(M-N). \tag{14}$$

Ungleich der Function Φ können dabei die Functionen ϕ, ψ ganz beliebig (mit gemeinsamem Existenzbereich) angenommen werden. Nur können selbstverständlich, im Falle sie identisch sind, nicht auch identische Lösungen σ, τ der dann einzigen Schwarz'schen Gleichung benutzt werden.

Die zu gegebenen Functionen ϕ, ψ gehörigen Curven sind natürlich nicht alle unter einander congruent. Vielmehr stehen je zwei unter ihnen in der Beziehung, dass ihre Tangentenindicatrices zu einander eigentlich-kreisverwandt sind. Congruent sind alle die Curven, die erhalten werden, wenn man die Grössen σ, τ derselben linearen Transformation unterwirft (§ 1).

Die Gleichung $\phi\equiv\psi$ kennzeichnet die Familie der *regulären Schraubenlinien*, der Curven nämlich, für die $R:T$ einen constanten Werth hat, also die krummen geodätischen Linien auf unebenen Cylinderflächen, oder die Curven, deren Tangentenindicatrices (reguläre oder parabolische) Kreise sind. Dass diese Curven durch Quadraturen bestimmt werden können, ergiebt sich hier aus der Thatsache, dass aus jeder Lösung der zugehörigen Riccati'schen Gleichung durch die Transformation (13) eine zweite abgeleitet werden kann. (R. Gl., § 6). Doch wird man hier, wie auch in anderen Fällen, näher liegende und elementarere Methoden vorziehen (vgl. § 8). Die Gleichung $M-N\equiv$ const. kennzeichnet die *Curven constanter Torsion*, insbesondere die Gleichung $M-N\equiv 0$ *die regulären ebenen Curven* — die zugleich Schraubenlinien sind —; die Gleichung $M+N\equiv$ const. ist charakteristisch für *die Curven constanter Krümmung*.

Ist die betrachtete reguläre Curve *relle*, so sind ihre Begleiter conjugiert-complex, und die Gleichungen $\phi\equiv\phi(s)$, $\bar{\psi}\equiv\bar{\psi}(\bar{s})$ drücken dieselbe analytische Abhängigkeit aus: Dieses ist die Bedingung dafür, dass unter den zu ϕ, ψ gehörigen regulären Curven sich reelle finden. Ist überdies ein *reeller*

Zug vorhanden, und ist $\{ s \equiv \bar{s} \}$ ein entsprechendes Werthgebiet des reellen Bogens, so müssen die Functionen ϕ und ψ, nach Potenzen von $s - s_0 \equiv \bar{s} - \bar{s}_0$ entwickelt, conjugiert-complexe Coefficienten haben.

§ 5. *Differentialinvarianten der regulären Curven.*

Wie im Falle der Minimalcurven, so hängt auch im Falle der regulären Curven das Problem der natürlichen Gleichungen mit der Theorie gewisser linearer Differentialgleichungen zusammen. Eine solche von der dritten Ordnung wird nämlich geliefert durch die lineare Relation, die immer zwischen den Vectoren x', x'', x''', x'''' stattfinden muss. Um ihre Coefficienten zu bestimmen, hat man die auf den Bogen bezogenen Semiinvarianten (234), (341), (412), (123), — die eben dadurch in Invarianten übergegangen sind — auszudrücken durch die Functionen Φ, Ψ und ihre Differentialquotienten. So entsteht, ähnlich wie in § 2, eine allgemeinere Frage, die durch den folgenden Lehrsatz beantwortet wird:

Benutzt man bei einer regulären Curve den Bogen als Parameter, so gehen dadurch die Semiinvarianten $(i|\kappa)$ *und* $(i\kappa l)$ *für* $i, \kappa, l \leqq n$ *in rationale Functionen von*

$$\Phi,\ \Phi',\ \Phi'',\ \ldots,\ \Phi^{(n-2)};\ \Psi,\ \Psi',\ \Psi'',\ \ldots,\ \Psi^{(n-3)}$$

über und zwar treten in den Nennern dieser Ausdrücke lediglich Potenzen der absoluten Invariante Φ *auf.*

Mit ebenso beschaffenen Coefficienten lassen sich alle Semicovarianten $(\kappa|\omega)$ *und* $(i\kappa\omega)$ *— für* $i < \kappa \leqq n$ *— linear ausdrücken durch*

$$(x'|\omega), \qquad (x''|\omega), \qquad (x'x''\omega).$$

Wir gehen wieder zunächst bis zum Werthe $n = 4$ und erhalten dann, wenn wir neben den Zeichen

$$\Phi \equiv (2|2)_s, \qquad \Psi \equiv (123)_s$$

noch die Abkürzung

$$\text{(1)} \qquad \mathrm{X} \equiv \frac{1}{\Phi}\left\{ \Phi^3 + \Psi^2 + \frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2 \right\}$$

gebrauchen, die folgende Tabelle, die die für die Anwendungen wichtigsten Formeln umfasst:

$$\begin{array}{lll} (1|1) \equiv 1, & *, & *, \\ (1|2) \equiv 0, & (2|2) \equiv \Phi, & *, \\ (1|3) \equiv -\Phi, & (2|3) \equiv \tfrac{1}{2}\Phi', & (3|3) \equiv \mathrm{X}, \\ (1|4) \equiv -\tfrac{3}{2}\Phi', & (2|4) \equiv \tfrac{1}{2}\Phi'' - \mathrm{X}, & (3|4) \equiv \tfrac{1}{2}\mathrm{X}', \end{array} \qquad \text{(2)}$$

$$(4|4) \equiv \frac{1}{\Phi}\left[\tfrac{9}{4}\Phi\Phi'\Phi' + \Psi'\Psi' + (\tfrac{1}{2}\Phi'' - \mathrm{X})^2\right],$$

. . . . . .

$$(2)\qquad \begin{aligned} &(123)\equiv\Psi, \qquad (134)\equiv\frac{1}{\Phi}\left\{\frac{\Phi'\Psi'-\Psi\Phi''}{2}+\Psi X\right\},\\ &(124)\equiv\Psi', \qquad (234)\equiv\tfrac{1}{2}\{2\Phi\Psi'-3\Psi\Phi'\}. \end{aligned}$$

Wir ersetzen jetzt $n(\geqq 4)$ durch $n-1$, und nehmen für diesen Werth den Satz als richtig an. Wir erhalten dann durch Differentiation alle Semiinvarianten der Form $(i\,|\,n)$ für $i<n$, also alle bilinearen Semiinvarianten mit Ausnahme von $(n\,|\,n)$. Ebenso entstehen durch Differentiation alle Semiinvarianten $(i\kappa n)$, wo $i<\kappa<n$, mit Ausnahme derer, für die $\kappa=n-1$ ist. Durch Bildung der Producte $(1\,|\,1)(i,\ n-1,\ n)$ für $i=2,\ 3,\ \cdots,\ n-2$ erkennt man sodann, dass alle diese Grössen sich als ganze Functionen der schon berechneten Ausdrücke und der einen Semiinvariante $(1,\ n-1,\ n)$ darstellen lassen. Die gleiche Eigenschaft aber hat das Product dieses letzten Differentialausdrucks mit $\Phi\equiv(2\,|\,2)$. Die noch fehlende Semiinvariante $(n\,|\,n)$ erhält man schliesslich, gleichfalls in der behaupteten Form, durch Bildung des Quadrates $(12n)^2$.

Der zweite Theil des aufgestellten Satzes ergiebt sich am einfachsten durch Bildung des Productes $(12n)(12\omega)$:

$$(3)\qquad \Phi\cdot(n\,|\,\omega)\equiv\Phi\cdot(1\,|\,n)\cdot(1\,|\,\omega)+(2\,|\,n)\cdot(2\,|\,\omega)+(12n)\cdot(12\omega).$$

$$(4)\qquad \Phi(mn\omega)\equiv\Phi\cdot(1mn)\cdot(1\,|\,\omega)+(2mn)\cdot(2\,|\,\omega)+(12\,|\,mn)\cdot(12\omega). \quad (m<n).$$

Wie die Herleitung zeigt, bilden die gefundenen Grössen ein in gewissem Sinne *vollständiges System absoluter* — in ihrer Gesammtheit nicht rationaler — *Differentialinvarianten und Differentialcovarianten* der betrachteten regulären Curve.

Zum Beispiel wird die Bedingung dafür, dass eine reguläre Curve im weiteren Sinne des Wortes sphärisch ist (uneben-sphärisch oder eben), allgemein ausgedrückt durch die Gleichung

$$2\Phi\Phi_s'\cdot\Psi_s'+(\Phi_s'\Phi_s'-2\Phi\Phi_s''+4\Psi^2)\Psi\equiv 0.$$

(Vgl. A. C., Nr. 60).

Auf dem hier vorgezeichneten Wege wird man, unter Anderem, auch die Bedingung dafür ermitteln können, dass die Tangentenindicatrix einer regulären Curve ein sphärischer Kegelschnitt oder wenigstens ein Stück einer solchen Curve ist. Die Differentialgleichung der Kegelschnitte ist in der hierfür nöthigen allgemeinen Form vom Verfasser aufgestellt worden.* Man hat darnach die in der Gleichung

$$\begin{aligned} &9\cdot(123)^2\cdot\{(126)+10\cdot(234)+5\cdot(135)\}-\\ &\qquad-45\cdot(123)\cdot(124)\cdot\{2\cdot(134)+(125)\}+40\cdot(124)^3\equiv 0 \end{aligned}$$

vorkommenden Differentialinvarianten durch Φ, Ψ und die Differentialquotienten

* Leipziger Berichte, Bd. 53, (1901) S. 349.

dieser Functionen (bis zu denen dritter Ordnung) auszudrücken.

Die auszuführende Rechnung ist natürlich umständlich. Im nächsten § werden wir deshalb die gesuchte Bedingung auf einem anderen Wege, und zwar mit ganz kurzer Rechnung ableiten. $(\Theta \equiv 0)$.

§ 6. *Über gewisse lineare Differentialgleichungen 3. Ordnung.*

Die Formel (3) in § 5 zeigt, dass im Falle einer *ebenen* regulären Curve der Differentialvector $y \equiv x'$ der linearen Differentialgleichung 2. Ordnung

$$y'' - \frac{1}{2}\frac{\Phi'}{\Phi}y' + \Phi y \equiv 0$$

genügt, deren allgemeine Lösung in der Form

$$c_1 \cos \int \sqrt{\Phi}\, ds + c_2 \sin \int \sqrt{\Phi}\, ds$$

enthalten ist. Wir schliessen diesen Fall von der ferneren Betrachtung aus, nehmen also an, dass die niederste lineare Differentialgleichung, der der Vector y genügt, von der dritten Ordnung ist,

$$y''' + 3Ay'' + 3By' + Cy \equiv 0 \qquad (\Psi \not\equiv 0). \tag{1}$$

Die Werthe der Coefficienten A, B, C können dann den Formeln des vorigen § entnommen werden:

$$\begin{aligned}
A &\equiv -\frac{1}{3}\frac{(124)}{(123)} \equiv -\frac{1}{3}\frac{\Psi'}{\Psi},\\
B &\equiv \frac{1}{3}\frac{(134)}{(123)} \equiv \frac{1}{3\Phi}\left\{ \mathrm{X} + \frac{\Phi'}{2}\frac{\Psi'}{\Psi} - \frac{\Phi''}{2} \right\},\\
C &\equiv -\frac{(234)}{(123)} \equiv -\frac{1}{2}\left\{ 2\Phi \cdot \frac{\Psi'}{\Psi} - 3\Phi' \right\},\\
\mathrm{X} &\equiv \frac{1}{\Phi}\left\{ \Phi^3 + \Psi^2 + \frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2 \right\}.
\end{aligned} \tag{2}$$

Wie die Ableitung zeigt, ist in dieser Form jede beliebige lineare homogene Differentialgleichung 3. Ordnung enthalten, bei der (mindestens) eine aus drei linear-unabhängigen Lösungen gebildete quadratische Form von nicht verschwindender Discriminante einen von Null verschiedenen constanten Werth hat.

Man wird dann nämlich drei solche Lösungen y_1, y_2, y_3 derart auswählen können, dass

$$y_1^2 + y_2^2 + y_3^2 \equiv 1 \tag{3}$$

wird: Diese Functionen y_1, y_2, y_3 der unabhängigen Veränderlichen s könne

sonach als die Coordinaten der Tangentenindicatrix einer regulären *unebenen* Curve aufgefasst werden.

Es entsteht nunmehr die Frage, durch was für Besonderheiten die Coefficienten A, B, C der hier auftretenden linearen Differentialgleichungen 3. Ordnung gekennzeichnet sind; wie man diese Coefficienten in die Form (2) setzen, und wie man schliesslich die Auflösung einer gegebenen Gleichung der Art bewirken kann.

Wir nehmen also an, dass die Gleichungen (2) bestehen. Aus den Gleichungen für A und C folgt dann

$$\Psi' \equiv -3A\Psi, \qquad \Phi' \equiv \tfrac{2}{3}C - 2A\Phi,$$
$$\Phi'' \equiv \tfrac{2}{3}C' - \tfrac{4}{3}AC + 2\{2A^2 - A'\}\Phi;$$

durch Substitution dieser Werthe in die Gleichung für B ergiebt sich

$$C^2 - 3\{C' + 3AC\}\Phi + 9\{A' + 2A^2 - 3B\}\Phi^2 + 9\{\Phi^3 + \Psi^2\} \equiv 0,$$

und hieraus, durch nochmalige Differentiation und Elimination von Φ', Ψ':

$$2AC^2 + \{C'' + AC' - A'C + 12BC - 14A^2C\}\Phi$$
$$- 3\{A'' - 3B' + 2C + 12AB - 8A^3\}\Phi^2 + 18A\{\Phi^3 + \Psi^2\} \equiv 0.$$

Aus den beiden letzten Gleichungen kann nunmehr die Summe

$$C^2 + 9\{\Phi^3 + \Psi^2\}$$

eliminiert werden.

Wir setzen zur Abkürzung

$$F \equiv A' + 2A^2 - 3B, \qquad G \equiv C' + 3AC, \tag{4}$$

so dass obige Gleichung die Form

$$C^2 - 3G\cdot\Phi + 9F\Phi^2 + 9\{\Phi^3 + \Psi^2\} \equiv 0 \tag{5}$$

annimmt, ferner

$$\Theta \equiv A'' + 6AA' + 4A^3 - 3B' - 6AB + 2C,$$
$$\mathrm{H} \equiv C'' + 7A\cdot C' + \{-A' + 4A^2 + 12B\}\cdot C,$$

oder

$$\Theta \equiv \frac{dF}{ds} + 2AF + 2C, \tag{6}$$

$$\mathrm{H} \equiv \frac{dG}{ds} + 4AG - 4CF, \tag{7}$$

und erhalten, als Eliminationsresultat, die Gleichung:

$$3\Theta\Phi - \mathrm{H} \equiv 0. \tag{8}$$

Aus der abgeleiteten linearen Differentialgleichung für Φ ergiebt sich schliess-

lich eine Gleichung fur Θ und H:

$$\Theta H' - H\Theta' \equiv 2\Theta \{ C\Theta - AH \}. \tag{9}$$

Dieser, die in den Differentialquotienten von A, B, C bis zur dritten Ordnung ansteigt, müssen also die Functionen A, B, C jedenfalls genügen. Das ist indessen nicht hinreichend. Damit der eingeschlagene Weg auch in umgekehrter Richtung gangbar sei, muss sich vielmehr ein nicht identisch verschwindender Werth von Φ, und hierauf — durch Lösung einer reinen quadratischen Gleichung — ein ebenfalls nicht identisch verschwindender Werth von Ψ berechnen lassen. Es muss also, wie die Gleichung (8) zeigt,

$$\textit{entweder} \quad \Theta \not\equiv 0, H \not\equiv 0 \quad \textit{oder} \quad \Theta \equiv 0, H \equiv 0$$

sein.

Im ersten Fall kann man Φ bestimmen: Es kommt also nur noch die weitere Einschränkung hinzu, dass der dann zu berechnende Werth von Ψ nicht identisch gleich Null wird.

Machen wir die zweite Annahme, so folgt aus $\Theta \equiv 0$, dass eine quadratische Form von y_1, y_2, y_3 identisch verschwindet,* und dass folglich *alle* quadratischen Formen von y_1, y_2, y_3, die einem gewissen Büschel angehören, constante Werthe haben. Umgekehrt liegt auch, wenn das eintritt, immer der zweite Fall vor. Es kann dann das genannte Büschel das Quadrat einer linearen Form enthalten, deren Werth natürlich ebenfalls constant sein muss. Die Bedingung hierfür ist $C \equiv 0$.

Wir werden hiernach, in der Theorie der hier in Betracht kommenden speciellen linearen Differentialgleichungen drei Hauptfälle unterscheiden, deren einige dann noch weiter zu gliedern sein werden. Auf diese Unterscheidung bezieht sich der folgende Lehrsatz, der durch das Vorgetragene begründet ist:

Wenn zwischen drei passend gewählten linear-unabhängigen Lösungen einer linearen homogenen Differentialgleichung 3. Ordnung eine Gleichung der Form (3) *besteht, so liegen folgende Möglichkeiten vor:*

I. *Es verschwindet keine quadratische Form von y_1, y_2, y_3 identisch. Der Ort des Punktes y ist dann eine sphärische Curve oder Stück einer solchen,† die uneben und auch verschieden ist von einem sphärischen Kegelschnitt.*

II. *Es verschwindet eine derartige Form, die vorgelegte Gleichung hat aber keine constante Lösung. Der Ort des Punktes y ist dann ein unebener sphärischer Kegelschnitt, oder Stück eines solchen.*

*Siehe z. B. PICARD, *Traité d'analyse*, t. III, p. 556, oder R. Gl., Nr. 42.

† Auch in der Theorie der linearen Differentialgleichungen ist es nicht üblich, die Begriffe Curve und Curvenstück zu unterscheiden. Man kann aber zuweilen Aussagen über den ganzen Verlauf, oder nahezu über den ganzen Verlauf einer Curve machen, und wir wünschen, wo dies der Fall ist, es hervortreten zu lassen. So entziehen sich der Parameterdarstellung $x \equiv x(s)$ höchstens isolierte Stellen, während auf der Curve (y) der Parameter s sehr wohl eine natürliche Grenze haben kann. Deshalb sagen wir im Texte "Curve oder Curvenstück," nicht einfach "Curve." Freilich ist auch diese Terminologie noch nicht ganz zufriedenstellend.

III. *Die vorgelegte Gleichung hat eine constante Lösung. Der Ort des Punktes y ist dann ein (regulärer oder parabolischer) Kreis, oder Stück eines solchen Kreises (der jedoch immer von einem Hauptkreis verschieden ist).*

Durch das Vorhergehende schon erledigt ist die erste dieser Annahmen:

Damit der Fall I *eintrete, ist nothwendig und hinreichend, dass die Functionen* Θ, H *der Gleichung* (9) *genügen, aber nicht identisch verschwinden, und dass ausserdem der Ausdruck*

$$C^2\Theta^3 - G\Theta^2\mathrm{H} + F\Theta\mathrm{H}^2 + \tfrac{1}{3}\mathrm{H}^3 \tag{10}$$

nicht identisch gleich Null ist.

Durch Berechnung der Functionen Φ, Ψ (Nr. 8, 5) *lässt sich dann die Integration der vorgelegten Gleichung zurückführen auf die Integration einer Schwarz'schen Differentialgleichung* (§ 3).

Soviel wird sich erreichen lassen, so lange nicht weitere Besonderheiten eintreten.*

§ 7. *Fortsetzung: Lineare Differentialgleichungen 3. Ordnung, die mit der Theorie der elliptischen Functionen zusammenhängen.*

Wir wenden uns jetzt zur Betrachtung des zweiten der in § 6 unterschiedenen Fälle, setzen also nunmehr voraus, dass $\Theta \equiv 0$, $\mathrm{H} \equiv 0$, aber $C \not\equiv 0$ ist. Dann werden sich in dem Büschel quadratischer Formen von y_1, y_2, y_3, die constante Werthe haben, entweder drei Formen von verschwindender Discriminante finden, oder deren zwei, oder eine, und jede von diesen muss einen nicht verschwindenden Werth haben. Die Form, die den Werth Null annimmt, hat eine nicht verschwindende Discriminante. Eine zweite Form von nicht verschwindender Discriminante kann willkürlich ausgewählt werden, und sie liefert dann die Gleichung der Einheitskugel, bei passender Wahl der Coordinaten in der üblichen Form

$$y_1^2 + y_2^2 + y_3^2 = 1. \tag{1}$$

Überdies wird man es erreichen können, dass die hinzutretende zweite Gleichung entweder die Form

$$\kappa_1 y_1^2 + \kappa_2 y_2^2 + \kappa_3 y_3^2 = 0 \qquad (\kappa_1 \neq \kappa_2 \neq \kappa_3 \neq 0) \tag{2a}$$

annimmt, oder zweitens die Form

$$y_2(y_2 + iy_3) = \text{const., besser}$$

$$y_1^2 + y_2^2 + y_3^2 + 2\kappa y_2(y_2 + iy_3) = 0 \qquad (\kappa \neq 0, -1), \tag{2b}$$

oder drittens die Form $y_1(y_2 + iy_3) = \text{const.}$, insbesondere z. B. $= -1$, wofür wir besser schreiben können

$$y_1^2 + y_2^2 + y_3^2 + 2y_1(y_2 + iy_3) = 0. \tag{2c}$$

* Diese Fälle lassen sich bis jetzt nicht erschöpfend angeben. R. Gl., Einleitung.

Das vorgelegte Problem nimmt also nunmehr eine präcisere Form an: Erstens werden Kriterien für diese drei Möglichkeiten zu finden sein, und zweitens Lösungstripel, die je eines der aufgestellten Gleichungspaare befriedigen. Zugleich hat sich ergeben:

Im Falle II *ist der Ort der Punkte y entweder eine — vollständige oder unvollständige — sogenannte elliptische Curve 4. Ordnung* (*a*), *oder eine rationale Curve 4. Ordnung mit Doppelpunkt auf dem absoluten Kegelschnitt* (*b*), *oder endlich eine Curve 4. Ordnung mit Spitze* (*c*).

Wir nehmen jetzt an, dass der Fall (*a*) vorliegt. Dann können wir die Gleichungen (1) und (2*a*) identisch befriedigen durch Einführung elliptischer Coordinaten:

$$(3)\quad y_1 \equiv \sqrt{\kappa_2 - \kappa_3}\sqrt{z - e_1},\quad y_2 \equiv \sqrt{\kappa_3 - \kappa_1}\sqrt{z - e_2},\quad y_3 \equiv \sqrt{\kappa_1 - \kappa_2}\sqrt{z - e_3},$$

indem wir nämlich drei Constante e_1, e_2, e_3 gemäss den Bedingungen

$$e_1 + e_2 + e_3 = 0,$$

$$(\kappa_2 - \kappa_3)e_1 + (\kappa_3 - \kappa_1)e_2 + (\kappa_1 - \kappa_2)e_3 = -1,$$

$$\kappa_1(\kappa_2 - \kappa_3)e_1 + \kappa_2(\kappa_3 - \kappa_1)e_2 + \kappa_3(\kappa_1 - \kappa_2)e_3 = 0$$

bestimmen:

$$e_1 = \frac{1}{3}\,\frac{\kappa_2(\kappa_3 - \kappa_1) - \kappa_3(\kappa_1 - \kappa_2)}{(\kappa_2 - \kappa_3)(\kappa_3 - \kappa_1)(\kappa_1 - \kappa_2)} \qquad \text{(etc.)}.$$

Setzen wir dann noch

$$(\kappa_2 - \kappa_3)e_2e_3 + (\kappa_3 - \kappa_1)e_3e_1 + (\kappa_1 - \kappa_2)e_1e_2 = e_0,$$

so bestehen die Ungleichungen

$$(4)\qquad e_0 \neq e_1 \neq e_2 \neq e_3,$$

und es lassen sich die Verhältnissgrössen $\kappa_1 : \kappa_2 : \kappa_3$ wieder durch die Verhältnissgrössen $e_0 : e_1 : e_2 : e_3$ ausdrücken:

$$\kappa_1 = -\frac{(e_2 - e_0)(e_3 - e_0)}{(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)} \qquad \text{(etc.)}.$$

Führen wir ferner die Abkürzung

$$\Pi \equiv \sqrt{-1}\cdot\sqrt{(e_0 - e_1)(e_0 - e_2)(e_0 - e_3)}$$

ein, so kann eine Abhängigkeit zwischen dieser Quadratwurzel und den Wurzelgrössen $\sqrt{\kappa_2 - \kappa_3}$, $\sqrt{\kappa_3 - \kappa_1}$, $\sqrt{\kappa_1 - \kappa_2}$ wie folgt erklärt werden:

$$\Pi = (e_2 - e_3)(e_3 - e_1)(e_1 - e_2)\cdot\sqrt{\kappa_2 - \kappa_3}\sqrt{\kappa_3 - \kappa_1}\sqrt{\kappa_1 - \kappa_2}.$$

Nunmehr kann man, von den Gleichungen (3) ausgehend, die Functionen Φ

und Ψ darstellen

$$(5) \qquad \Phi \equiv (y'|y') \equiv -\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^2 \cdot (z-e_0),$$

$$(6) \qquad \Psi \equiv (yy'y'') \equiv -\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^3 \cdot \Pi,$$

ferner aber auch die Coefficienten der linearen Differentialgleichung 3. Ordnung, der die Functionen y_1, y_2, y_3 genügen:

$$(7) \quad \begin{aligned} A &\equiv -\frac{1}{3}\frac{(yy'y''')}{(yy'y'')} \equiv -\frac{d}{ds}\log\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}, \\ B &\equiv \frac{1}{3}\frac{(yy''y''')}{(yy'y'')} \equiv -\frac{1}{3}\frac{z'''}{z'} + \\ &\quad + \frac{d}{ds}\log\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}} \cdot \frac{z''}{z'} + 3\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^3 \cdot z, \\ C &\equiv -\frac{(y'y''y''')}{(yy'y'')} \equiv -\frac{3}{2}\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^2 \cdot z'. \end{aligned}$$

Diese Gleichungen (7) liefern nun das vollständige System der Bedingungen, denen die Functionen A, B, C zu genügen haben, und sie lassen sich dann auch unschwer nach e_1, e_2, e_3, z auflösen (die durch die weitere Gleichung $e_1 + e_2 + e_3 = 0$ verbunden sind). Am einfachsten geht man wohl so zu Werke, dass man zwischen s und z noch eine zur Lösung des Integrationsproblems übrigens nicht nöthige Hülfsveränderliche u einschiebt — das zu der Curve (y) gehörige Integral 1. Gattung* — vermöge der Substitutionen:

$$(8) \qquad z \equiv \wp(u), \qquad \frac{dz}{du} \equiv \wp'(u) \equiv -2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}.$$

Die Gleichungen (7), deren System wir sogleich noch etwas ergänzen wollen, nehmen dann, bei Gebrauch der schon eingeführten Abkürzungen

$$(9) \qquad F \equiv A' + 2A^2 - 3B, \qquad G \equiv C' + 3AC$$

eine übersichtlichere Form an:

$$(10) \quad \begin{aligned} A &\equiv -\frac{u''}{u'}, \qquad F \equiv 3\wp(u)\cdot(u')^2, \\ B &\equiv -\frac{1}{3}\frac{u'''}{u'} + \left(\frac{u''}{u'}\right)^2 - \wp(u)\cdot(u')^2, \\ C &\equiv -\tfrac{3}{2}\wp'(u)\cdot(u')^3, \qquad G \equiv -\tfrac{3}{2}\wp''(u)\cdot(u')^4, \end{aligned}$$

* Wir entnehmen hier der Theorie der elliptischen Functionen lediglich einige übliche Zeichen.

oder also

$$\frac{ds}{du} \equiv -e^{\int A ds}, \qquad \wp(u) \equiv \tfrac{1}{3} F \cdot e^{2\int A ds},$$
$$\wp'(u) \equiv \tfrac{2}{3} C \cdot e^{3\int A ds}, \qquad \wp''(u) \equiv -\tfrac{2}{3} G \cdot e^{4\int A ds}. \tag{11}$$

Hieraus, und aus den in (8) enthaltenen Gleichungen

$$\{\wp'(u)\}^2 \equiv 4\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}$$

oder

$$\{\wp'(u)\}^2 \equiv 4\wp^3(u) - g_2\wp(u) - g_3,$$
$$\wp''(u) \equiv 4\wp^2(u) - \tfrac{1}{2} g_2, \qquad \wp'''(u) \equiv 12\wp(u)\wp'(u)$$

folgen nun wieder die auf anderem Wege schon abgeleiteten Bedingungsgleichungen

$$\Theta \equiv F' + 2AF + 2C \equiv 0, \qquad H \equiv G' + 4AG - 4CF \equiv 0. \tag{12}$$

Nehmen wir umgekehrt an, dass neben $C \not\equiv 0$ diese Identitäten bestehen, so erhalten wir aus (10) oder (11) zunächst drei Gleichungen zur Bestimmung von e_1, e_2, e_3, nämlich:

$$e_1 + e_2 + e_3 = 0$$
$$g_2 \equiv -4\{e_2e_3 + e_3e_1 + e_1e_2\} \equiv \tfrac{4}{3}\{F^2 + G\} e^{4\int A ds} \tag{13}$$
$$g_3 \equiv 4e_1e_2e_3 \equiv -\tfrac{4}{27}\{2F^3 + 3FG + 3C^2\} e^{6\int A ds}.$$

Zufolge (12) werden dann diese Grössen constant. Es liegt aber kein Grund dafür vor, dass e_1, e_2, e_3 von einander verschieden ausfallen sollten. Es ergiebt sich also noch eine weitere, eben für den behandelten Fall (a) charakteristische Bedingung:

$$\tfrac{1}{16}\{g_2^3 - 27g_3^2\} e^{-12\int A ds} \equiv \tfrac{1}{27}\{4(F^2+G)^3 - (2F^3+3FG+3C^2)^2\} \not\equiv 0. \tag{14}$$

Ist auch diese Voraussetzung erfüllt, so erhält man schliesslich durch Einsetzen der Function

$$z \equiv \tfrac{1}{3} F e^{2\int A ds} \tag{15}$$

in die Wurzelgrössen $\sqrt{z - e_\varkappa}$ drei linear-unabhängige Lösungen der gegebenen Gleichung. Es ist nämlich, wie aus $\Theta \equiv 0$, d. h. $F' + 2AF \equiv -2C$ und $C \not\equiv 0$ folgt, die aus (15) berechnete Function z nicht constant. Wie dann, nach Annahme einer passend gewählten Constanten e_0, die verlangten Particularlösungen y_1, y_2, y_3 gefunden werden können, wurde schon oben gezeigt.

Aber die in der Formel (15) noch vorkommende Quadratur ist überflüssig. Wir denken uns in dem Integral $\int A ds$ der unteren Grenze einen bestimmten

Werth s_0 beigelegt, und schreiben gleichzeitig die Constante e_0 etwas anders:

$$\int Ads = \int_{s_0}^{s} Ads, \qquad e_0 = e_0^* e^{\int_{s_0}^{s_1} Ads}.$$

Dann leuchtet ein, dass bei Verschiebung von s_0 die Functionen

$$e_0,\ e_1,\ e_2,\ e_3 \text{ und } z$$

alle denselben constanten Factor annehmen, so dass y_1^2, y_2^2, y_3^2 ganz ungeändert bleiben und y_1, y_2, y_3 ebenfalls ungeändert bleiben können.

Die gefundenen Formeln enthalten daher, entsprechend dem Umstande, dass nach Festlegung der Einheitskugel in dem mehrfach genannten Büschel das System der Functionen y_1^2, y_2^2, y_3^2 eindeutig bestimmt sein muss, nur einen wesentlichen Parameter e_0^*. Insbesondere ergiebt sich, dass man, je nach Umständen, eine der drei Constanten $g_3 : g_2, g_2, g_3$ willkürlich vorschreiben und so die in (15) vorkommende Exponentialgrösse bestimmen kann. Man hat dazu entweder die Gleichung

$$(16,\ \alpha) \qquad e^{2\int Ads} \equiv -\frac{9(F^2+G)}{2F^3+3FG+3C^2}\cdot\frac{g_3}{g_2},$$

oder, wenn nämlich

$$2F^3+3FG+3C^2 \equiv 0, \qquad g_3 = 0$$

ist, die Gleichung

$$(16,\ \beta) \qquad e^{2\int Ads} \equiv \tfrac{1}{2}\sqrt[2]{\frac{3g_2}{F^2+G}}$$

oder endlich, wenn $F^2+G \equiv 0$, $g_2 = 0$ ist, die Gleichung

$$(16,\ \gamma) \qquad e^{2\int Ads} \equiv -\tfrac{3}{2}\sqrt[3]{\frac{2g_3}{2F^3+3F^2G+3C^2}}$$

zu benutzen.

Der Fall II*a*) *liegt also dann vor, wenn zu den Bedingungen* $C \not\equiv 0$, $\Theta \equiv 0$, $H \equiv 0$ *die Ungleichung* (14) *tritt. Die Auflösung der gegebenen Gleichung erfordert dann lediglich algebraische Operationen.*

Dass von den eingeführten Wurzelgrössen keine durch einen rationalen Ausdruck ersetzt werden kann, so lange nicht weitere (in unseren Voraussetzungen nicht enthaltene) Besonderheiten vorliegen, lässt sich unschwer einsehen.

Zur Bestimmung des auf der Curve (y) überall endlichen Integrals

$$(17) \qquad u \equiv \int ds \cdot e^{-\int Ads}$$

braucht man nach dem Vorgetragenen nur die Differentialgleichung für y zu kennen, und man erhält dann diese Function durch Quadratur über gewisse Wurzelgrössen.

In der vorausgehenden Darlegung haben wir darauf Rücksicht genommen, die den Formeln innewohnenden Symmetrie-Eigenschaften hervortreten zu lassen. Will man vielmehr auf den Fall reeller Curven und namentlich derer mit reellen Zügen besondere Rücksicht nehmen, so sind einige der getroffenen Festsetzungen entsprechend zu ändern.

Etliche der abgeleiteten Formeln lassen sich noch etwas weiter entwickeln, wenn man $e_0 = \wp(v)$ setzt ($v \not\equiv 0$, ω_1, ω_2, ω_3 modd. $2\omega_1$, $2\omega_2$) und dann zu Ξ- oder Θ-Functionen übergeht.

§ 8. *Fortsetzung: Grenzfälle.*

Die Betrachtung der Fälle II*b*) und II*c*) wird wesentlich erleichtert durch die Bemerkung, dass im vorigen § die mit den Formeln (5) einsetzende Entwickelung grossentheils unabhängig davon ist, ob die Grössen e_1, e_2, e_3 von einander verschieden sind, oder nicht. Nur bleiben die auf diese Art ermittelten Lösungen in den Grenzfällen nicht linear-unabhängig. Die fehlenden Lösungen aber kann man unschwer durch Grenzübergänge erhalten.

Im Falle II*b*), *der durch die Bedingungen* $C \not\equiv 0$, $\Theta \equiv 0$, $H \equiv 0$ *und*

$$(1) \qquad 4(F^2 + G)^3 - (2F^3 + 3FG + 3C^2)^2 \equiv 0, \qquad F^2 + G \not\equiv 0$$

gekennzeichnet wird, setzen wir $e_2 = e_3$, und dem entsprechend

$$e_1 = 2\eta, \quad e_2 = -\eta, \quad e_3 = -\eta, \quad \zeta \equiv \frac{z}{\eta}.$$

Wir erhalten dann, wenn ζ aus der Gleichung

$$(2) \qquad \zeta \equiv \frac{z}{\eta} \equiv \frac{1}{3\eta} F e^{2\int A dt} \equiv -\frac{2F(F^2 + G)}{2F^3 + 3FG + 3C^2}$$

bestimmt wird, für ζ einen nicht constanten Werth (da $F \equiv 0$ und auch schon $F' + 2AF \equiv 0$, ebenso wie $G \equiv 0$, das Verschwinden von C zur Folge haben würden). Die Ausdrücke

$$(3) \qquad (\zeta - 2)^{\frac{1}{2}}, \quad (\zeta + 1)^{\frac{1}{2}}, \quad (\zeta + 1)^{-\frac{1}{2}}$$

stellen dann ein System von drei linear-unabhängigen Lösungen dar. Dieses wird leicht durch ein System y_1, y_2, y_3 von der verlangten Form (§ 7, Nr. 2*b*) ersetzt:

$$y_1 \equiv \lambda\sqrt{\zeta - 2}, \quad y_2 \equiv \frac{1}{2\kappa\lambda} \frac{1}{\sqrt{\zeta + 1}}, \quad y_3 \equiv i\left\{\lambda\sqrt{\zeta + 1} + \frac{1}{2\kappa\lambda} \frac{1}{\sqrt{\zeta + 1}}\right\},$$

$$\lambda = \sqrt{-\frac{\kappa + 1}{3\kappa}}.$$

Im Falle II*c*) treten zu den Bedingungen $C \not\equiv 0$, $\Theta \equiv 0$, $H \equiv 0$ die

Gleichungen

$$F^2 + G \equiv 0, \qquad 2F^3 + 3FG + 3C^2 \equiv 0.$$

Diese ziehen aber die Gleichungen $\Theta \equiv 0$, $H \equiv 0$ nach sich. In der That ist jetzt nothwendig $F \not\equiv 0$, $G \not\equiv 0$; man kann daher

$$\Xi \equiv -\frac{C}{F} \equiv \frac{1}{3}\frac{G}{C}$$

setzen, woraus

$$A \equiv \Xi - \frac{\Xi'}{\Xi}, \qquad F \equiv 3\Xi^2,$$

$$(5) \qquad B \equiv \frac{1}{3}\left\{\frac{d}{ds}\left(\Xi - \frac{\Xi'}{\Xi}\right) + 2\left(\Xi - \frac{\Xi'}{\Xi}\right)^2 - 3\Xi^2\right\},$$

$$C \equiv -3\Xi^3, \qquad G \equiv -9\Xi^4$$

und $\Theta \equiv 0$, $H \equiv 0$ folgt. *Die Bedingungen dieses Falles sind also*

$$(6) \qquad C \not\equiv 0, \quad F^2 + G \equiv 0, \quad F^3 - 3C^2 \equiv 0;$$

man kann alle hierher gehörigen linearen Differentialgleichungen 3. Ordnung mit Hülfe einer nicht constanten, sonst aber völlig willkürlichen Function $\Xi(s)$ bilden.

Wird, entsprechend den Entwickelungen des vorigen Paragraphen,

$$z \equiv \tfrac{1}{3} F e^{2\int A ds}, \quad \text{und} \quad Z = \sqrt{z}$$

gesetzt, so folgt

$$(7) \qquad Z \equiv e^{\int \Xi ds},$$

und man erhält in den Ausdrücken

$$(8) \qquad Z, \quad Z^{-1}, \quad Z^{-3}$$

ein System von drei linear-unabhängigen Lösungen der vorgelegten Gleichung. Die in (7) vorkommende Quadratur ist nicht zu entbehren, so lange die willkürliche Function $\Xi(s)$ nicht specialisiert wird. Ein System von drei Lösungen y_1, y_2, y_3, die der in §7 aufgestellten Forderung (2c) oder also den Gleichungen

$$(9) \qquad y_1^2 + (y_2 + iy_3)(y_2 - iy_3) \equiv 1, \qquad 2y_1(y_2 + iy_3) \equiv -1,$$

genügen, ist

$$(10) \quad y_1 \equiv \frac{i}{2}\frac{1}{Z}, \qquad y_2 \equiv \frac{i}{2}\left\{Z - \frac{1}{Z} - \frac{1}{4Z^3}\right\}, \qquad y_3 \equiv \frac{1}{2}\left\{Z + \frac{1}{Z} + \frac{1}{4Z^3}\right\}.$$

Man kann aber jetzt die Forderung (9) offenbar auf unendlich viele Weisen erfüllen.

Es hat sich also ergeben:

Im Falle II*b*) *erfordert die Lösung der vorgelegten Gleichung nur algebraische Operationen, im Falle* II*c*) (*höchstens und in der Regel*) *eine Quadratur.*

Wichtiger als diese stets imaginären Grenzfälle sind die der Annahme $C \equiv 0$ entsprechenden.

Setzen wir also jetzt voraus, dass $C \equiv 0$, $\Theta \equiv 0$, sei — woraus $H \equiv 0$ folgt — so ergiebt die Gleichung $C \equiv 0$, dass

$$\Phi \equiv \Xi^2, \qquad \Psi \equiv \kappa\Xi^3 \qquad (\kappa = \text{const.}, \neq 0) \tag{11}$$

gesetzt werden kann. Es folgt dann

$$A \equiv -\frac{\Xi'}{\Xi}, \qquad F \equiv -(1+\kappa^2)\Xi^2, \qquad G \equiv 0,$$
$$B \equiv \frac{1}{3}\left\{-\frac{\Xi''}{\Xi} + 3\left(\frac{\Xi'}{\Xi}\right)^2 + (1+\kappa^2)\Xi^2\right\}; \tag{12}$$

man kann wieder alle hierher gehörigen Gleichungen mit Hülfe einer nicht identisch verschwindenden, aber sonst willkürlichen Function $\Xi(s)$ bilden.

Sind die Coefficienten A, B gegeben, so sind zwei Fälle zu unterscheiden:

$$C \equiv 0, \qquad \Theta \equiv 0, \qquad F \not\equiv 0, \tag{13a}$$

$$C \equiv 0, \qquad F \equiv 0. \tag{13b}$$

Im ersten Falle kann man der Constanten κ einen geeigneten Werth nach Belieben beilegen, und dann Ξ bestimmen:

$$\Xi \equiv \frac{1}{\sqrt{1+\kappa^2}}\sqrt{-F} \qquad (\kappa \neq 0, i, -i). \tag{14a}$$

Ein System von drei linear-unabhängigen Lösungen, die den hier etwa zu stellenden Forderungen

$$y_1^2 + y_2^2 + y_3^2 \equiv 1, \qquad y_1 \equiv \text{const.} \qquad (\text{const.} \neq 0, 1, -1) \tag{15a}$$

genügen, ist gegeben durch die Formeln

$$y_1 = \frac{\kappa}{\sqrt{1+\kappa^2}}, \qquad y_2 \equiv \frac{1}{\sqrt{1+\kappa^2}}\cos\int\sqrt{-F}ds,$$
$$y_3 \equiv \frac{1}{\sqrt{1+\kappa^2}}\sin\int\sqrt{-F}ds. \tag{16a}$$

Die geometrische Bedeutung dieses Formelsystems und die Unentbehrlichkeit der ausgeführten Quadratur sind evident. Für die zugehörigen Curven $x \equiv \int yds$ hat man

$$\frac{R}{T} \equiv \pm\kappa \qquad (\kappa \neq 0, i, -i); \tag{17a}$$

diese Curven sind Schraubenlinien, die auf gewöhnlichen unebenen Cylindern geodätische Linien sind.

Ist dagegen $F \equiv 0$, so folgt $\kappa = \pm i$, und Ξ wird durch Quadratur ermittelt:

$$\Xi \equiv e^{-\int A ds} \tag{14b}$$

Man kann dann auf unendlich viele Arten — deren einzelne noch von zwei willkürlichen Parametern abhängt — ein System von Lösungen y_1, y_2, y_3 herstellen, die den Forderungen

$$y_1^2 + y_2^2 + y_3^2 = 1, \qquad y_2 + iy_3 = 1 \tag{15b}$$

genügen. Man kommt zu derartigen Formeln am schnellsten, wenn man unter den auf der Einheitskugel verlaufenden parabolischen Kreisen — die alle zu einander schon vermöge automorpher Bewegungen der Einheitskugel congruent sind — einen geeigneten auswählt,

$$y_1 \equiv -ip, \qquad y_2 \equiv 1 + \tfrac{1}{2}p^2, \qquad y_3 \equiv \frac{i}{2}p^2,$$

und dann die Veränderliche p, die für den parabolischen Kreis ein natürlicher Parameter ist (A. C., § 5), als eine zu suchende Function von s behandelt. Man erhält

$$p \equiv i\int \Xi ds \equiv i\int ds\, e^{-\int A ds},$$

$$y_1 \equiv \int \Xi ds, \qquad y_2 \equiv 1 - \frac{1}{2}\left\{\int \Xi ds\right\}^2, \qquad y_3 \equiv -\frac{i}{2}\left\{\int \Xi ds\right\}^2. \tag{16b}$$

Die zugehörigen Schraubenlinien $x \equiv \int y ds$ entsprechen der Gleichung

$$\frac{R}{T} \equiv \pm i. \tag{17b}$$

Sie sind also geodätische Linien auf (unebenen) Cylindern mit Minimalgeraden als Erzeugenden.

Die Annahme $A \equiv 0$ kennzeichnet in beiden Fallen (a), (b) die (unebenen) *gemeinen* Schraubenlinien, die entweder (a) transcendent oder (b) rational, sogenannte Lion'sche Curven (dritter Ordnung) sind.

Es hat sich also ergeben:

In den durch die Bedingungen (13) *gekennzeichneten Fällen* IIIa) *und* IIIb) *verlangt die Integration der vorgelegten Gleichung höchstens eine* (a) *oder zwei* (b) *Quadraturen.*

Dass diese auch im Falle (b) nicht entbehrlich sind, so lange nicht noch weitere Annahmen hinzukommen, ist evident.

Das in § 6 *gestellte Problem ist hiermit vollständig gelöst. Damit zugleich sind die natürlichen Gleichungen der regulären Curven hergestellt worden,*

deren Tangentenindicatrix ein sphärischer Kegelschnitt ist; es sind die Bedingungen ermittelt, unter denen eine durch ihre natürlichen Gleichungen gegebene Curve dieser Familie angehört, und es ist ferner auch die Integration dieser Gleichungen geleistet.

Leicht lassen sich nunmehr auch, mit gleichartigem Ergebniss, die natürlichen Gleichungen der regulären Curven behandeln, deren Haupt- oder Binormalenindicatrix ein spärischer Kegelschnitt ist. Wir gehen hierauf nicht mehr ein.

Schlussbemerkungen.

Die in § 6–§ 8 behandelten linearen Differentialgleichungen gehören zu denen, auf die die von Herrn E. *Vessiot* begründete Integrationstheorie Anwendung findet.* In der That stimmen die von uns ausgeführten Operationen *der Art nach* genau überein mit denen, die sich bei Anwendung der Grundsätze des genannten Autors ergeben würden. Wir haben indessen von dieser schönen Theorie mit Absicht keinen Gebrauch gemacht. Einmal wollten wir die an den Leser zu stellenden Anforderungen nicht höher spannen, als eben nöthig war. Dann aber wird eine möglichst zweckmässige Führung der Rechnungen im concreten Falle durch die allgemeine Theorie noch nicht gewährleistet. Endlich hat Herr *Vessiot* den Gegenstand gar nicht berührt, der im Mittelpunkte unserer Untersuchung steht: den Zusammenhang gewisser linearer Differentialgleichungen mit der Theorie der natürlichen Gleichungen analytischer Curven, und die Zurückführung des einen Integrationsproblems auf das andere. In den hier betrachteten besonderen Fällen hat daher unsere elementare Methode den reicheren Inhalt.

Wir erwähnen schliesslich noch, dass der bekannte Zusammenhang der in § 2 gelegentlich betrachteten linearen Differentialgleichungen 3. Ordnung mit der Theorie der linearen Differentialgleichungen 2. Ordnung (R. Gl., § 12) unschwer auf die Gleichungen

$$y''' + 3Ay'' + 3By' + Cy \equiv 0$$

des in § 6–§ 8 betrachteten Typus ausgedehnt werden kann, wenn auch das Resultat minder einfach zu sein scheint.

Nachdem nämlich auf die beschriebene Art die Functionen Φ, Ψ und aus diesen die Functionen ϕ, ψ bestimmt worden sind, kann man die mit $\sigma(s)$, $\tau(s)$ bezeichneten Functionen in die Form

$$\sigma \equiv \frac{v_1}{w_1}, \qquad \tau \equiv \frac{v_2}{w_2}$$

setzen, wo v_1, w_1 und v_2, w_2 passend gewählte Lösungen von je einer der

*Annales de l'École Normale Supérieure, ser. III, t. 9 (1892).

Gleichungen

$$u_1'' + \frac{\phi}{4} u_1 \equiv 0, \qquad u_2'' + \frac{\psi}{4} u_2 \equiv 0$$

sind. (R. Gl., § 10). Wie die Gleichungen (2) in § 4 zeigen, hat man dann in den Quotienten

$$\frac{v_1 v_2}{v_1 w_2 - v_2 w_1}, \qquad \frac{w_1 w_2}{v_1 w_2 - v_2 w_1}, \qquad \frac{v_1 w_2 + v_2 w_1}{v_1 w_2 - v_2 w_1}$$

drei linear-unabhängige Lösungen der vorgelegten Gleichung 3. Ordnung.

Ob zwischen der Riccati'schen Gleichung

$$\frac{d\Omega}{ds} \equiv \frac{i}{2}\left\{\frac{1}{T} - 2\frac{\Omega}{R} - \frac{\Omega^2}{T}\right\},$$

zu der die classische Theorie der natürlichen Gleichungen führt, und den von uns benutzten unter einander äquivalenten Gleichungen

$$\frac{d\mu}{ds} \equiv \frac{\phi}{4} + \mu^2, \qquad \frac{dv}{ds} = \frac{\psi}{4} + v^2$$

ein ähnlich-einfacher Zusammenhang stattfindet, wie der, den wir in § 2 in einem Grenzfall nachgewiesen hatten, das wissen wir nicht zu sagen.

NEAPEL, 5. *Nov.* 1909.

TWO-DIMENSIONAL CHAINS AND THE ASSOCIATED COLLINEATIONS IN A COMPLEX PLANE*

BY

JOHN WESLEY YOUNG

Introduction.

The recognition of the abstract identity of geometry and analysis, which results from the notion of coördinates on the one hand and from the classical work of VON STAUDT † on the algebra of throws on the other, and which recent work on the foundations of geometry has fully established, has brought with it a broader conception of the content of geometry. It has meant not only the introduction of imaginary elements and the resulting conception of a complex space (of any number of dimensions), but it has also led to the consideration of geometries with respect to any number-system (finite or infinite), i. e., of spaces the elements of which may be determined by sets of numbers (coördinates) belonging to a given number-system. ‡ An important result of the recognition of the identity referred to is the emphasis it places on the possibility of using geometric or synthetic methods in the solution of analytic problems. It seems likely that the fact that such methods have received comparatively little attention hitherto has resulted in a loss of power. The present paper, it is hoped, will tend to substantiate this assertion.

We are here concerned with certain fundamental problems in the projective geometry on a *complex* plane, i. e., a plane the points of which are determined by sets of homogeneous coördinates (x_1, x_2, x_3), where the x_i are any ordinary complex numbers. not all zero. Though the investigation is essentially geometric and the results are susceptible of immediate application to problems of importance in geometry, these results are of even greater interest in the theory of functions of two complex variables.

To make this clear we shall glance briefly at the corresponding problems on a complex line, the results of which are well-known in the theory of functions

* Presented to the Society, April 25, 1908, under a slightly different title.

† VON STAUDT, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856–60, Heft 2, p. 261.

‡ Cf., e. g., § 2 of the paper by Professor VEBLEN and myself cited below; and for the finite cases, O. VEBLEN and W. H. BUSSEY, *Finite projective geometries*. Transactions of the American Mathematical Society, vol. 7 (1906), pp. 241–59.

of one complex variable and which I have recently considered from the point of view of projective geometry.* The notion of a *linear* (or one-dimensional) *chain* on a complex line was introduced by VON STAUDT† in connection with his introduction of imaginary elements into geometry and has since been made fundamental in recent work on the foundations of geometry.‡ Such a chain may be defined as any class of points on a line which is projective with the class of real points on a line. There is on a line one and only one chain containing any three distinct points of the line.§ The place of this notion in the theory of functions of a complex variable becomes apparent, if we adopt the usual representation of complex numbers (i. e., of the points of our line) by the real points of a plane or sphere. The chains on the line are then represented by the real circles (and straight lines) of the plane or sphere, and the study of the projectivities on the line (i. e., of the linear fractional transformations on the complex variable) with reference to their behavior toward the chains on the line is fundamental in the projective geometry on the line as well as in the theory of functions. This study leads to the classification of the projectivities into hyperbolic, elliptic, parabolic and loxodromic and to the well-known systems of chains (circles) associated with the first three of these types of projectivities.

In the present paper we are concerned with the corresponding problems in the plane. Here the notion of a *planar* (or two-dimensional) *chain* is fundamental. A planar chain may be conveniently described as any class of points and lines in the plane which may be obtained from the class of real points and lines by a projective collineation. A planar chain is then a two-dimensional spread of points and lines within the four-dimensional spread of all points and lines in our complex plane.‖ It follows readily from the description given that there is one and only one planar chain through any four points of the plane, no three of which are collinear.¶ The notion of a planar chain was first introduced

* J. W. YOUNG, *The geometry of chains on a complex line*, Annals of Mathematics, series 2, vol. 11 (1909), No. 1, pp. 33–48. This paper is referred to in the sequel by the letter *A*.

† VON STAUDT, loc. cit., p. 137.

‡ Cf. PIERI, *Nuovi principii di geometria projettiva complessa*, Memorie della R. Accademia delle Scienze di Torino, series II, vol. 45 (1905), pp. 189–235, in which the chain is chosen as one of the primitive notions; also O. VEBLEN and J. W. YOUNG, *A set of assumptions for projective geometry*, American Journal of Mathematics, vol. 30 (1908), pp. 347–380, where the chain is defined in terms of order and continuity relations.

§ The coördinate system on a line is determined as soon as 0, 1, ∞ are assigned to three distinct points of the line, which are entirely arbitrary. The chain determined by the three points is then simply the class of all points on the line that have real coördinates.

‖ In this paper we use the word "dimension" throughout in the sense of *real* dimension, i. e., a spread of n dimensions is one whose elements may be made to depend on n independent *real* parameters. The plane is, of course, a spread of two *complex* dimensions, but we shall not use the word here in this sense.

¶ We may indeed establish a coördinate system in the plane by choosing any three points

by JUEL and SEGRE* in connection with their study of anti-projective transformations.† These planar chains are of value in those problems in the plane in which the distinction is maintained between real and imaginary elements. The projective point of view, however, emphasizes the fact that this distinction is relative, since any planar chain may be taken as the real planar chain; the points and lines not belonging to this chain are then "imaginary" with reference to the points and lines of the chain.

In § 1 below we define a planar chain geometrically in terms of the notion of a linear chain and derive certain properties of these chains which are needed in the later developments. In § 2 we determine necessary and sufficient conditions that a collineation in the plane leave a planar chain invariant; this is equivalent to the determination of the conditions that a collineation may be represented (transformed into one) with real coefficients. We are thus led to certain characteristic *systems of planar chains* which form a generalization of the systems of circles referred to above. In the Conclusion these systems are briefly considered and reference is made to certain immediate applications of the results obtained. Here also is outlined the important problem of the order relations in a complex plane, to which I expect to return on a future occasion.

§ 1. *Definition and fundamental properties of planar chains.*

We assume the theorems of alignment in the ordinary complex projective geometry of the plane.‡ All points and lines considered are coplanar. We assume further the notion and fundamental properties of (linear) chains on a line. We then define a planar chain as follows:

(which are not collinear) as the vertices of the triangle of reference and assigning to any fourth point which is not on a side of this triangle the coördinates (1, 1, 1). The planar chain determined by these four points then consists of all points and lines the ratios of whose coördinates are real numbers.

*JUEL, *Über einige Grundgebilde der projectiven Geometrie*, Acta Mathematica, vol. 14 (1890), pp. 1–30; SEGRE, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia delle Scienze di Torino, vol. 25 (1890), pp. 276–301, 430–457; vol. 26 (1890), pp. 35–71, 592–612. Though closely related in subject matter, the present paper has little direct contact with either of the two papers cited.

†The term "anti-projective" is due to SEGRE and seems to have been generally adopted. JUEL calls the anti-projective transformations in the plane "Symmetralitäten." He describes a planar chain in his introduction substantially as we have described it above (though in his description the planar chain consists of a class of points only). SEGRE on the other hand obtains them as the class of invariant elements of certain anti-projective transformations. It may be noted in passing that the property of a planar chain to the effect that the point of intersection of two lines joining two pairs of points of the chain is a point of the chain, which JUEL apparently takes as his definition of a planar chain (loc. cit., p. 3) is not sufficient to characterize it. The property is characteristic of any *planar field*, i. e., of any set of points and lines in a plane whose coördinates are numbers of a given field. The notion of a general field in a geometry of any number of dimensions I defined in a paper presented to the Society (Chicago section), January 2, 1909.

‡Our developments are subject, e. g., to Assumptions *A*, *E*, *H*, *C* and *I* of the paper by Professor VEBLEN and myself quoted above.

DEFINITION. Given two linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$, on distinct lines and having a point O in common, a point is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it is the intersection of two lines, each of which joins a point of $\mathfrak{C}_1$ to a distinct point of $\mathfrak{C}_2$. A line is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it joins two points linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$. The class of all points and lines linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$ is called the *two dimensional chain* or the *planar chain* determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$.

If a scale * be established on the line of $\mathfrak{C}_1$ by choosing the point O as 0 and any other two distinct points of $\mathfrak{C}_1$ as 1 and ∞ respectively, the points of $\mathfrak{C}_1$ correspond to the set of all ordinary real numbers (and ∞) and only these. If then a scale be established in a similar manner on the line of $\mathfrak{C}_2$, the coördinate system in the plane determined by these two scales will assign to every point (or line) of the planar chain $\mathfrak{C}^2$ determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$ a set of real coördinates; and, conversely, to every set of real coördinates will correspond a point (or line) of $\mathfrak{C}^2$. Since a projective collineation in the plane is equivalent to a change in the system of coördinates, this justifies the analytic definition suggested in the introduction:

Any class of points and lines which can be obtained from the real points and lines of the plane by a projective collineation is a planar chain.

Since a projective transformation transforms linear chains into linear chains, it follows at once from the definition given above (and also from the analytic characterization just obtained), that

I. *Any class of points and lines of a plane which is projective with a planar chain is a planar chain.* †

The following fundamental *internal* properties of a planar chain $\mathfrak{C}^2$ may be deduced readily (they are simply properties of the points and lines of a *real* projective plane):

II. *There is one and only one planar chain containing four points* (*lines*) *of a plane no three of which are collinear* (*concurrent*).

III. *Any two lines of a planar chain meet in a point of the chain.*

IV. *Any line of a planar chain has the points of a linear chain in common with the planar chain, and only these.*

We should note also the *principle of duality* in a planar chain, where the elements "point" and "linear chain" are dual elements.

The following *external* properties are of fundamental importance:

VI. *Every line of the plane which is not a line of a planar chain* $\mathfrak{C}^2$ *in the plane has one and only one point in common with* $\mathfrak{C}^2$.

This theorem is most readily proved analytically. Let a system of homo-

* VEBLEN and YOUNG, loc. cit., § 2, p. 353.

† In this theorem the projective transformation may be either a collineation or a reciprocity. The theorem is still true, moreover, if the transformation in question is *anti-projective* (cf. SEGRE, *Un nuovo campo etc.*, loc. cit.).

geneous coördinates be established in such a way that the points of $\mathfrak{C}^2$ correspond to *real* ratios $x_1 : x_2 : x_3$ of the coördinates. The equation of any line in the plane which is not a line of $\mathfrak{C}^2$ is then $u_1x_1 + u_2x_2 + u_3x_3 = 0$, where the ratios $u_1 : u_2 : u_3$ are not all real. Placing $u_j = u_j' + iu_j''$ ($j = 1, 2, 3$; u_j', u_j'' real) the equations

$$u_1'x_1 + u_2'x_2 + u_3'x_3 = 0, \qquad u_1''x_1 + u_2''x_2 + u_3''x_3 = 0,$$

determine a real point $x_1 : x_2 : x_3$ of the line. The line cannot have more than one point in common with $\mathfrak{C}^2$, for otherwise it would be a line of $\mathfrak{C}^2$.

The argument dual to this gives at once:

VII. *Through every point of the plane which is not a point of a planar chain $\mathfrak{C}^2$ in the plane passes one and only one line of $\mathfrak{C}^2$.*

These theorems lead at once to the notion of pairs of points (lines) *inverse* or *conjugate* with respect to a planar chain. Let P be any point not on a given planar chain $\mathfrak{C}^2$ and let l be the line of $\mathfrak{C}^2$ through P (VII). This line l has a linear chain in common with $\mathfrak{C}^2$ (IV); the point $\bar{P}$ which is the inverse of P with respect to this linear chain is called the *inverse* or *conjugate* of P with respect to the planar chain $\mathfrak{C}^2$; P, $\bar{P}$ form a pair of *conjugate points* with respect to $\mathfrak{C}^2$. The latter phrase is justified by the evident fact that either of two such points is the conjugate of the other. Analytically two conjugate points with respect to the planar chain all of whose elements have real coördinates are determined by coördinates of which one set consists of the conjugate complex numbers of those forming the other set. Every point of a planar chain $\mathfrak{C}^2$ is, by definition, its own conjugate with respect to $\mathfrak{C}^2$.*

The process dual to the one described gives rise to the notion of pairs of *conjugate lines* with respect to a planar chain. It follows further that *the line joining two points conjugate with respect to a planar chain $\mathfrak{C}^2$ is a line of $\mathfrak{C}^2$; and the point of intersection of two lines conjugate with respect to $\mathfrak{C}^2$ is a point of $\mathfrak{C}^2$.* It is then at once evident analytically that

VIII. *The line joining two points A, B is conjugate with respect to a planar chain with the line joining the two points $\bar{A}$, $\bar{B}$ which are conjugate with A, B respectively; and every point on a line l is conjugate with a point on the line conjugate with l.*

Theorems I–VIII just given are sufficient to derive the fundamental properties of the geometry of chains in a plane. In preparation for the application of these properties to the classification of collineations in the plane with reference to their behavior toward the planar chains of the plane and the derivation of the resulting characteristic systems of planar chains, we call attention to a few general results of importance.

* The transformation obtained by replacing each point of the plane by its conjugate with respect to a planar chain (wherein the points of the planar chain are self-conjugate) is called by SEGRE (loc. cit.) an *antinvolution* with double elements. Cf. the footnote, p. 282.

We note first, that a triangle in the plane determines an infinite system of chains containing the vertices and sides of this triangle; there is, in fact, through every point of the plane which is not on a side of the triangle one and only one chain of this system, and one and only one through every line which does not meet one of the vertices (II). In general, two planar chains of this system have only the vertices and sides of the triangle in common; certain pairs of chains, however, have more than these elements in common, as the following considerations will show. A linear chain $\mathfrak{C}$ of points and a point not on $\mathfrak{C}$ determine an infinite system of planar chains each of which contains P and the points of $\mathfrak{C}$; viz., there is one such chain through any point (distinct from P and not in $\mathfrak{C}$) of a line joining P to any point of $\mathfrak{C}$. For there are thus determined two linear chains ($\mathfrak{C}$ and one on the line through P and containing P) which determine a planar chain (Definition). *Two distinct planar chains, however, can not have more points in common than a linear chain $\mathfrak{C}$ of points and a point P not on $\mathfrak{C}$.* For, let Q be another common point; there exist then under the hypothesis two points A, B of $\mathfrak{C}$ such that no three of the points A, B, P, Q are collinear. Hence, the two planar chains would coincide (II).* In other words *two planar chains which have the points of a linear chain and two other points in common coincide.*

It should be noted, however, that a linear chain of points and two points not on this chain do not in general determine a planar chain in which they lie. They will determine a planar chain only if the line joining the two points meets the linear chain in question (IV).

Another important system of chains is determined as follows:

DEFINITION. Any planar chain is said to be *about* two points B, C, if B, C are conjugate points with respect to the planar chain.

Now let A, B, C be the vertices of a triangle. The class of all planar chains through A and about B, C is the system in question. There is one and only one chain of this system through any point not on a side of the triangle ABC. For, let P be such a point; the line AP meets the line BC in a point Q. Let $\mathfrak{C}$ be the linear chain through Q and about B, C. The chain $\mathfrak{C}$ and the two points P, A determine a unique planar chain with the specified property. Two chains of this system have in general only the point A and the line BC in common; certain pairs of these chains have, however, also a linear chain of points on BC and a linear chain of lines through A in common.

* Two distinct chains through the points of a linear chain C and a point P not on C have also a linear chain of lines through P in common (the linear chains of points on these lines are, of course, different for the two planar chains).

§ 2. *Collineations with invariant planar chains.*

We now seek necessary and sufficient conditions that a collineation in the plane leave a planar chain $\mathfrak{C}^2$ invariant.* Suppose π is a collineation leaving a planar chain $\mathfrak{C}^2$ invariant, and let a system of homogeneous coördinates be determined so that the elements of $\mathfrak{C}^2$ are determined by real ratios of the coördinates. It is then clear that a necessary and sufficient condition that π leave $\mathfrak{C}^2$ invariant is that when π is represented analytically by equations

$$x_i' = \Sigma a_{ij} x_j \qquad (i, j = 1, 2, 3),$$

the coefficients a_{ij} shall be real numbers (or at least that they may be rendered real by multiplication with a suitably chosen factor of proportionality). We seek, however, a simpler and geometric condition. The five well-known types of collineations in the plane will be treated successively.

Type I. *Three (and only three) double points.* If the collineation be represented analytically as described above, the characteristic equation will have real coefficients and three distinct roots. There are then two cases to consider:

h) The roots of the characteristic equation are all real;

e) One of the roots is real and the other two conjugate complex.†

Type Ih. If the roots of the characteristic equation are all real the double points are all real. In other words, any invariant planar chain $\mathfrak{C}^2$ contains all three double points, A, B, C. Each of the sides of the triangle ABC then has a linear chain in common with $\mathfrak{C}^2$ (IV) and, since these lines are invariant, these linear chains are invariant under the projectivities on the invariant lines. Since these linear chains contain the (distinct) double points of the projectivities mentioned, *the projectivities on the sides of the invariant triangle must all be hyperbolic.*‡ This necessary condition is also sufficient. For let π be any collineation of Type I such that the projectivities on two sides AB and BC of the invariant triangle are hyperbolic, and let $\mathfrak{C}^2$ be *any* planar chain through A, B, C. The linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$ on AB, BC respectively are then invariant, since a hyperbolic projectivity on a line leaves every chain through the double points invariant (A, Theorem 14). Since $\mathfrak{C}^2$ is determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$, it follows that $\mathfrak{C}^2$ is invariant. That the linear chain of $\mathfrak{C}^2$ on CA is likewise invariant then follows almost immediately. Hence,

* That such collineations exist is obvious. Let A, B, C, D be the vertices of any quadrangle, and let $\mathfrak{C}^3$ be the planar chain determined by A, B, C, D. Let A', B', C', D' be any other four points of $\mathfrak{C}^3$ no three of which are collinear. The collineation determined by the homologous pairs A, A'; B, B'; C, C'; D, D' then clearly leaves $\mathfrak{C}^2$ invariant, since it transforms $\mathfrak{C}^3$ into the planar chain determined by A', B', C', D', which is identical with $\mathfrak{C}^2$ (II).

† These two cases are also readily obtained by a purely synthetic argument. Cf. footnote on next page.

‡ In accordance with the definition given in *A*, p. 42, the involutions are to be considered as both hyperbolic and elliptic.

THEOREM 1. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through the double points invariant is that the projectivities on two sides of the invariant triangle be hyperbolic. The projectivity on the third side is then likewise hyperbolic, and the collineation leaves every planar chain through the double points invariant.*

A collineation satisfying the conditions of this theorem we will call *hyperbolic of Type I*, or *of Type Ih.*

Moreover, a hyperbolic collineation of Type I leaves invariant no other planar chains than those mentioned in the theorem. For, if A is a double point, there is at least one line through A which meets a given invariant planar chain in a linear chain $\mathfrak{C}_1$ (VII). If this linear chain is invariant, A must be a point of C_1 and hence of the planar chain, since the projectivity in this chain is hyperbolic. If $\mathfrak{C}_1$ is not invariant, it is transformed into another linear chain of the invariant planar chain, and the lines of these two linear chains determine A as a point of the invariant planar chain. Hence,

THEOREM 2. *The system of invariant planar chains of a hyperbolic collineation of Type I consists of all the planar chains through the double points and only these. There is one chain of the system, and only one, through every point of the plane which is not on a side of the invariant triangle.*

Type Ie. If one of the roots of the characteristic equation is real and the other two are conjugate complex, any invariant planar chain $\mathfrak{C}^2$ contains one double point, say A, and the other two double points B, C are conjugate with respect to the planar chain.* The line BC is then a line of the planar chain (it also contains one invariant line, therefore; so that the invariant figure is self-dual within the planar chain) and has an invariant linear chain $\mathfrak{C}_1$ in common with the invariant planar chain (IV). Since $\mathfrak{C}_1$ does not contain the double points of the projectivity on BC, this projectivity must be elliptic.

This condition is not, however, sufficient. A necessary and sufficient condition is readily obtained by considering the projectivities on the two conjugate lines AB and AC. If P is any point of $\mathfrak{C}^2$ not on a side of the invariant triangle, and the line PC meets AB in a point M, the line PB meets AC in a point N which is the conjugate of M with respect to $\mathfrak{C}^2$; for the line conjugate with MC passes through B (the conjugate of C) and P, and must contain the conjugate of M which is on AC (VIII). Let P' be the point homologous with P and let the points M', N' on AB, AC be constructed. If $\mathfrak{C}^2$ is invariant,

* That any invariant planar chain of a collineation of Type I either passes through all the double points or passes through one and about the other two may be seen synthetically as follows. Suppose an invariant planar chain $\mathfrak{C}^2$ does not contain the double point B. Through B passes a line l of $\mathfrak{C}^2$. The line l must then be invariant; for otherwise B would be the intersection of two lines of $\mathfrak{C}^2$ and would hence be a point of $\mathfrak{C}^2$. The invariant linear chain of l is therefore about the two double points B and C (say) on l. No other line of $\mathfrak{C}^2$ can pass through B or C. Through the third double point A, however, passes a line of $\mathfrak{C}^2$, and since (as just noted) this line cannot be invariant, A is a point of $\mathfrak{C}^2$.

the projectivities on AB and AC must then be so related that if a pair of points are homologous under the projectivity on AB, their conjugates with respect to $\mathfrak{C}^2$ must be homologous under the projectivity on AC. Two projectivities which are related in this way with respect to a planar chain we will call *conjugate*. A necessary condition that $\mathfrak{C}^2$ be invariant under a collineation with double points A, B, C is then that the projectivities on AB and AC be conjugate with respect to $\mathfrak{C}^2$. But this condition is at once seen to be also sufficient. Hence,

THEOREM 3. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through one double point A and about the other two B, C invariant is that the projectivities on the two conjugate invariant lines be conjugate. The projectivity on the third side of the invariant triangle is then elliptic. The collineation leaves every planar chain through A and about B, C invariant.**

A collineation satisfying the condition of Theorem 3 we will call *elliptic of Type I* or *of Type Ie*. The double points on the side of the invariant triangle on which the projectivity is elliptic we will call the *conjugate* double points. We then have:

THEOREM 4. *The system of invariant planar chains of an elliptic collineation of Type I consists of all the planar chains about the conjugate double points and through the remaining double point, and only these. There is one and only one chain of this system through every point which lies in the plane but not on a side of the invariant triangle.*

Type II. *Two and only two double points.* The invariant figure consists of two points A, B, the line $l = AB$, and a line l' through A. The collineation is completely determined by the projectivity on the line l and the parabolic projectivity on l'. We note first that the characteristic equation of any collineation of this type has a double root, and that hence all the roots of this equation must be real. It then follows at once that *every invariant planar chain of such a collineation must contain the two points A and B and a linear chain of l'.*†

Let $\mathfrak{C}^2$ be any invariant planar chain. The linear chain of $\mathfrak{C}^2$ on AB is

* The theorem that *if a collineation of Type I leaves one planar chain* $\overset{\text{through the double points } A, B, C}{\text{through one double point } A \text{ and}}$ *about the other double points B, C invariant, it leaves every chain* $\overset{\text{through } A, B, C}{\text{through } A \text{ and about } B, C}$ *invariant*, may also be proved directly as follows: Let π be the given collineation and $\mathfrak{C}^2$ a planar chain which it leaves invariant. Let P be any point of $\mathfrak{C}^2$ which is not on a side of the invariant triangle of π and let P_1 be any other point in the plane not on a side of this invariant triangle. Let π_1 be the collineation which has the same invariant triangle as π and which transforms P into P_1. π_1 then clearly transforms $\mathfrak{C}^2$ into a planar chain $\mathfrak{C}_1^2$ through P_1. Moreover, if $\mathfrak{C}^2$ contains the double points A, B, C of π, so also will $\mathfrak{C}_1^2$; and if $\mathfrak{C}^2$ is through A and about B, C, so also is $\mathfrak{C}_1^2$. Now the transformation $\pi_1\pi\pi_1^{-1}$ clearly leaves $\mathfrak{C}_1^2$ invariant. But since π, π_1 have the same invariant triangle, they are commutative and we have $\pi_1\pi\pi_1^{-1} = \pi$; i. e., π leaves $\mathfrak{C}_1^2$ invariant.

† This result, as well as the corresponding one under Type III below, may also be established readily by a purely synthetic argument.

invariant under the projectivity on AB; hence this projectivity must be *hyperbolic*. Now let P be any point not on l or l'. We show that one and only one invariant planar chain of the collineation contains P. Let the line BP meet l' in a point Q and let Q' be the point homologous with Q under the parabolic projectivity on l'. Any invariant planar chain through P must contain Q (III) and hence Q'. The points P, B, Q', A determine uniquely a planar chain; hence, there is no more than one invariant planar chain through P. That the planar chain thus determined is indeed invariant follows from the fact that it is uniquely determined by its two linear chains on the line $l = AB$ and l' (Def.) and each of these linear chains is invariant; the one on l because it contains the two double points of a hyperbolic projectivity (A, Theorem 14), the one on l' because it contains the double point and one pair of homologous points of a parabolic projectivity (A, Theorem 15). We have then the following:

THEOREM 5. *A necessary and sufficient condition that a collineation of Type II leave a planar chain invariant is that the projectivity on one of the invariant lines be hyperbolic. The collineation then leaves every planar chain through the point B and an invariant linear chain on l' invariant.*

THEOREM 6. *The system of invariant planar chains is determined by the point B and the system of invariant chains of the parabolic projectivity on l'. There is one and only one chain of this system through any point not on l or l'.*

A collineation of Type II satisfying the condition of Theorem 5 may be called *hyperbolic of Type II* or *of Type IIh*.

Type III. *One and only one double point.* The invariant figure consists of a point A and a line l through A. The collineation is completely determined when the parabolic projectivity (with double point A) on l and one pair of homologous points (not on l) are given. As before, by reference to the characteristic equation which in this case has a triple root, we see that every invariant planar chain of such a collineation must contain A and a linear chain on l. Now let P, P' (not on l) be any pair of points homologous under the collineation in question. The line PP' then meets l in a point Q distinct from A. Let Q' be the point homologous with Q. Let $\mathfrak{C}^2$ be the planar chain determined by the linear chain $\mathfrak{C}$ through A, Q, Q' and the points P, P'. We show that $\mathfrak{C}^2$ is invariant under the collineation. We note first that the chain $\mathfrak{C}$ is invariant (A, Theorem 15). Similarly, the linear chain of lines at A determined by the lines l, AP, AP' is invariant, since it contains the double line and a pair of homologous lines of the parabolic projectivity in the pencil of lines at A. The line AP' is therefore transformed by the collineation into a line AM where M is a point of $\mathfrak{C}^2$ (III). The point P'' which is homologous with P' is now determined as the intersection of the lines $P'Q'$, AM, and is therefore a point of $\mathfrak{C}^2$ (III). Hence $\mathfrak{C}^2$ is invariant. Clearly also $\mathfrak{C}^2$ is the only invariant planar chain through P. We have then

THEOREM 6. *Every collineation of Type III leaves invariant one and only one planar chain through each point not on l. Every invariant planar chain contains an invariant linear chain of the parabolic projectivity on the double line; through every invariant linear chain of this parabolic projectivity there pass an infinite number of invariant planar chains of the collineation.*

A collineation of Type III may conveniently be called a *parabolic* collineation.

Type IV. *A point O and every point of a line l not through O is invariant.* A collineation of this type is called a *homology*, the point O and the line l being the *center* and *axis* respectively of the homology. A homology is completely determined when the center, axis, and one pair of homologous points (collinear with O) are given. Any invariant planar chain of a homology contains O and a linear chain on l. For let P, P' and Q, Q' be any two pairs of homologous points on any invariant planar chain (P, P', Q not collinear). The lines PP', QQ' meet in O which is therefore a point of the planar chain (III). The dual argument shows similarly that l contains two points and hence a linear chain of any invariant planar chain. Let $\mathfrak{C}^2$ be any planar chain through O and a linear chain $\mathfrak{C}$ on l. Any line joining O to a point of $\mathfrak{C}$ has a linear chain in common with $\mathfrak{C}^2$, and this linear chain must be invariant if $\mathfrak{C}^2$ is invariant. Hence, if a homology leaves a planar chain invariant, the projectivities along the invariant lines through O are *hyperbolic*. This necessary condition is also clearly sufficient, since the projectivities on all lines through O are hyperbolic, if one is, and a hyperbolic projectivity leaves every linear chain through its double points invariant. We have then

THEOREM 7. *A necessary and sufficient condition that a homology leave a planar chain invariant is that the projectivity on an invariant line through the center be hyperbolic. If this condition is satisfied, the homology leaves every planar chain through the center and any linear chain on the axis invariant.*

A homology satisfying the condition of this theorem we call a *hyperbolic homology* or a collineation of *Type IVh*.

THEOREM 8. *The system of invariant chains of a hyperbolic homology consists of all planar chains through the center and any linear chain on the axis, and of these only. Through any point not on the axis and distinct from the center there pass an infinite number of chains of this system. Through any two such points not collinear with the center passes one and only one chain of this system.**

*On every line through O there is a projectivity which has the center O and a point on the axis l for double points. It is well known, moreover, that any two projectivities on such lines are equivalent, i. e., can be transformed into each other (or have the same characteristic cross ratio). We may therefore classify homologies with reference to these projectivities into *hyperbolic*, *elliptic*, and *loxodromic*. The hyperbolic homologies alone leave a planar chain invariant. The elliptic homologies, however, leave invariant every linear chain about the double points on any line through A. The loxodromic homologies leave no planar chain invariant and no linear chain on any line except l. The elliptic and loxodromic projectivities leave invariant, however, every *system* of planar chains through A and a linear chain on l.

Type V. *All points of a line l and all lines through a point O on l are invariant.* A collineation of this type is called an *elation*, the line l and point O being the *axis* and *center* respectively. An elation is completely determined when the center, axis, and one pair of homologous points are given. If P, P' and Q, Q' are any two pairs of homologous points (not all collinear), the lines PP' and QQ' meet in O and the lines PQ and $P'Q'$ meet in a point of l. It follows at once that any invariant planar chain must contain O and a linear chain on l (III). The construction for the elation shows also at once that every planar chain containing a pair of homologous points and a linear chain on l through O is invariant. We have then

THEOREM 9. *Every elation leaves invariant every planar chain containing a pair of homologous points and a linear chain on the axis and through the center; and only these. Through any point not on the axis pass an infinite number of invariant planar chains; through two such points not collinear with the center passes one and only one.*

We note finally the following theorem which is an immediate consequence of the results of the preceding discussion:

THEOREM 10. *If a collineation leaves invariant more than one planar chain through a point which is not on a double line it is a perspective collineation, i. e., either a homology or an elation.*

§ 3. *Conclusion.*

Our discussion has led to the definition of six fundamental systems of planar chains in the plane, which form a natural generalization of the three well-known systems of linear chains on a line; i. e., of the three systems of circles (in the Argand representation of a complex variable) associated with the hyperbolic, elliptic, and parabolic transformations (linear fractional) on one complex variable. Each of the first four of these six systems, i. e., those associated with a hyperbolic collineation of Type I, an elliptic collineation of Type I, a hyperbolic collineation of Type II, or a parabolic collineation (Type III), consists of ∞^2* planar chains, one and only one through each point which is not a double point nor on a double line. Each of the last two systems, i. e., those associated with a hyperbolic homology or an elation, consists of ∞^4 planar chains, one and only one passing through two points neither of which is a double point and both of which are not on the same invariant line. In view of their importance, we shall describe these systems independently of the collineations by means of which they were obtained. They are as follows:

System Ih. Such a system is defined by any three points which are not collinear, and consists of all the planar chains through these points. Associated

* By ∞^2 is meant that the system may be made to depend on two independent *real* parameters; similarly for the ∞^4 in the next sentence, and similar expressions following.

with any such system is a two-parameter group of hyperbolic collineations having the three points as double points, each collineation of which leaves every chain of the system invariant.

System Ie. Three such systems are defined by any three points which are not collinear; each of these systems consists of all the chains through one of the points and about the other two. Associated with any system of this type is a two-parameter group of elliptic collineations having the three points as double points; each collineation of the group leaves every chain of the associated system invariant.

System II. Such a system consists of all the planar chains through two given points A, B and meeting a given line l' through A in a system of linear chains all mutually tangent at A. Associated with any such system is a two-parameter group of hyperbolic collineations, each collineation of which leaves every chain of the system invariant.

System III. Such a system meets a given line in a system of linear chains all mutually tangent at a given point A of the line, and meets the pencil of lines at A in a system of linear chains (of lines) all mutually tangent at the given line. Associated with any such system is a three-parameter group of parabolic collineations, each collineation of which leaves every chain of the system invariant.

System IV. Such a system consists of all the planar chains through a given line and a given point not on that line. Associated with any such system is a one-parameter group of hyperbolic homologies, each homology of which leaves every chain of the system invariant.

System V. Such a system consists of all the planar chains through a given line and through a given point not on that line. Associated with any such system is a one-parameter group of elations, each elation of which leaves every chain of the system invariant.

The results of this paper may be applied at once to the enumeration and geometric definition of *all* the continuous groups of collineations in the plane which leave a planar chain invariant. This is equivalent to the enumeration of all continuous groups of collineations in the plane which can be represented with real coefficients. No complete list of the groups of *real* collineations in a plane, in which the groups are characterized geometrically or in which the finite equations of the collineations are given, has ever been published, so far as the writer is aware.* The enumeration in both of these forms by the methods developed above is very simple, and offers a good example of a problem apparently analytic which yields readily to synthetic treatment.†

* LIE has given the infinitesimal equations of these groups. Cf. LIE-ENGEL, *Theorie der Transformationgruppen*, Leipzig (1893), vol. 2, pp. 78–109 and pp. 380–384.

† The results of this enumeration were presented by the writer to the Society September 11, 1908, and will be published in the near future.

We close with a word regarding the relation of our developments to the theory of functions of two complex variables. In the theory of functions of one complex variable the circles in the Argand plane owe their importance in part to the fact that the necessary and sufficient condition that a linear fractional transformation on the variable may be transformed into one with real coefficients is that it leave a circle invariant. In all analogous problems in the theory of functions of two complex variables the planar chains will play a role similar to that of the circles (linear chains) in the one-dimensional case. On the other hand, the circles owe their importance in the theory of functions of one complex variable also in part to the fact that they divide the points of the Argand plane into two regions. This aspect of the linear chain does not find its generalization in the planar chain, since a two-dimensional spread of points can not divide the four-dimensional spread of points in the complex plane into two regions. The generalization here in question may be found by combining a system of ∞^1 planar chains (or a system of ∞^2 linear chains) into a spread of three dimensions with the desired property. A three-dimensional spread of points dividing the complex plane into two regions is obtained very simply, moreover, as the class of all the points on the lines of a linear chain of lines. To this aspect of the problem and in general to the question of the order relations in a complex plane, I shall return on a future occasion. To this problem the present paper forms a preliminary investigation.

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GROUPS OF RATIONAL TRANSFORMATIONS IN A GENERAL FIELD*

BY

LEWIS IRVING NEIKIRK

Introduction.

Groups of linear transformations of a single variable of both finite and infinite orders are well known, but the only known examples of non-linear rational transformation groups in one variable are those given by the following writers: HERMITE, BETTI, and others have investigated special quantics, known as substitution quantics, with coefficients taken with respect to a prime modulus (p), which define substitutions on a set of residues (mod p) and generate finite groups (mod p). Substitution quantics with coefficients in a Galois field have been investigated by DICKSON in his dissertation,† where the reader will find a complete bibliography of the subject.

The object of this paper is to find all *non-linear* groups of rational transformations of a single variable. It is proved in § 1 that these groups of transformations define substitution groups on the roots of an equation $f(x) = 0$. They are a two-fold generalization of substitution quantics and form finite groups $(\text{mod } f(x))$. Section 2 is devoted to finding these transformations and section 3 to the conditions for the existence of such transformations in a general field F. The other articles apply and extend these results.

§1. *General developments.*

Consider a group G of rational integral transformations

$$T_i \equiv [x : \phi_i(x)],$$

$$\phi_i(x) = \sum_{j=0}^{j=m_i} \alpha_{ij} x^{m_i - j} \qquad (\alpha_{i0} \neq 0),$$

where the coefficients α_{ij} are elements of a general field F and the quantity x belongs to a set X_i in a field F' containing F. It is assumed that at least one m_i exceeds unity, so that the group is not linear.

* Presented to the Society (Chicago), April and December, 1909.

† L. E. DICKSON, *The analytical representation of substitutions on a power of a prime number of letters*, etc., Annals of Mathematics, ser. 1., vol. 11 (1896), pp. 65–120, 161–183.

Let $T_i(X_i) = X'_i$. Then *

(a) $X_i \equiv X'_i$, for every i.

(b) $X_i \equiv X_{i'} \equiv X$, for every i and i'.

(a) Since T_i^2 is in G, X'_i is a subset of X_i, and since T_i^{-2} is in G, X_i is a subset of X'_i. Therefore $X_i \equiv X'_i$.

(b) X_i must be a subset of $X_{i'}$ since $T_{i'}T_i$ is in G, and $X_{i'}$ must be a subset of X_i since $T_iT_{i'}$ is in G. Therefore $X_i \equiv X_{i'} \equiv X$.

Since T_i, of degree $m_i > 1$, has an inverse in G, let $T_i^{-1} = T_{i'}$. Then

$$T_iT_{i'} \equiv [x : x] = [x : \phi_i\{\phi_{i'}(x)\}],$$

whence

(1) $$\phi_i\{\phi_{i'}(x)\} = x,$$

so that x satisfies an equation of degree $m_im_{i'} > 1$, the leading coefficient being $\alpha_{i0}\alpha_{i'0} \neq 0$.

Therefore the elements of the set X are roots of an equation rational in F.

Let $X = (x_1, x_2, x_3, \cdots, x_n)$ be a set whose elements are the roots of an equation,

$$f(x) = \sum_{r=0}^{r=n} a_r x^{n-r} = 0,$$

with the coefficients in F and having no double root.

All the transformations reduce (mod $f(x)$) to degree $n-1$ or less.†

Let T_i change X according to the scheme

$$\begin{pmatrix} x_1 x_2 \cdots x_n \\ x_{i_1} x_{i_2} \cdots x_{i_n} \end{pmatrix}.$$

If any root is repeated in the lower line, T_i will not have an inverse in the group G. Therefore the lower line is a permutation of the upper line and T_i defines a substitution on the roots of $f(x) = 0$. Hence we have proved

THEOREM I. *The only non-linear groups of rational integral transformations on one variable are finite groups taken modulo $f(x)$ which define substitution groups on the roots of the equation $f(x) = 0$.*‡

§ 2. *Determination of the transformation corresponding to a given substitution.* §

Given a substitution on the roots of $f(x) = 0$,

$$S_i = \begin{pmatrix} x_1 x_2 \cdots x_n \\ x_{i_1} x_{i_2} \cdots x_{i_n} \end{pmatrix},$$

* BURNSIDE (*Theory of Groups*, p. 12) makes use of property (a) without explicit mention in the proof that if A_{-1} is the inverse of A, then A is the inverse of A_{-1}.

† H. WEBER, *Lehrbuch der Algebra*, vol. I, p. 170.

‡ The actual existence of these groups will be established in the next two articles.

§ L. E. DICKSON, Dissertation, l. c.

we seek the corresponding transformation T_i. We have the n linear equations

$$x_{i_t} = \phi_i(x_t) = \sum_{j=0}^{j=n-1} \alpha_{ij} x_t^{n-1-j} \qquad (t = 1, 2, \cdots, n)$$

between the n coefficients α_{ij}. From these

$$(2) \qquad \alpha_{ij} = \frac{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_{i_1} & x_1^{n-j-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_{i_2} & x_2^{n-j-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_{i_n} & x_n^{n-j-2} & \cdots & 1 \end{vmatrix}}{\pm\sqrt{\Delta}} \qquad (j = 0, 1, 2, 3, \cdots, n-1)$$

where Δ is the discriminant of $f(x)$, so that

$$\pm\sqrt{\Delta} = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix} \neq 0.$$

We can also determine T_i by the Lagrangian interpolation formula

$$\phi_i(x) = \sum_{t=1}^{t=n} \frac{x_{i_t} f(x)}{(x - x_t) f'(x_t)}, \qquad f(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

The coefficients of ϕ_i determined by either of these two methods are not necessarily contained in the general field F.

§ 3. *Condition for transformations with coefficients in F.*

THEOREM II. *The necessary and sufficient condition for the existence of the transformation T with coefficients in the field F on the roots of the equation $f(x) = 0$ with coefficients in F is that the substitution S be permutable with every substitution of the Galois group of $f(x) = 0$ for F.*

Let

$$S = \begin{pmatrix} x_t \\ x_{tS} \end{pmatrix} \qquad (t = 1, 2, \cdots, n).$$

Determine $\phi(x)$ by means of one of the two methods given in section 2. We have the equations

$$(3) \qquad x_{tS} = \phi(x_t) \qquad (t = 1, 2, 3, \cdots, n).$$

(1) Proof that condition is necessary. The coefficients of ϕ are in F by hypothesis. Hence we may apply to (3) the substitutions R of the Galoisian group.* Hence

$$x_{tSR} = \phi(x_{tR}).$$

But, by (3),

$$x_{tRS} = \phi(x_{tR}).$$

Hence $x_{tRS} = x_{tSR}$ for every t, and thus $RS = SR$.

(2) Proof that the condition is sufficient. By hypothesis, $RS = SR$ for every R in the Galoisian group.

Let $x_{tR} = x_p$. Then $x_{tSR} = x_{tRS} = x_{pS}$. Hence if R replaces x_t by x_p it replaces x_{tS} by x_{pS}. In § 2, $x_{1S}, \cdots, x_{nS}$ were denoted by $x_{i_1}, \cdots, x_{i_n}$. Hence if R replaces x_t by x_p, it replaces x_{i_t} by x_{i_p}. Hence the coefficients of ϕ given by equation (2) are unaltered by R and thus belong to F.

§ 4. *The representation of substitutions.*

The substitution

$$S_i \equiv \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix}$$

can be represented by the transformation

$$T_i \equiv [x_t : x_{\phi_i(t)}],$$

where

$$\phi_i(t) = \sum_{j=0}^{j=n} \frac{i_j f(t)}{(t-j)f'(j)}, \qquad f(t) = (t-1)(t-2)\cdots(t-n).$$

We may also determine the coefficients of

$$\phi_i(t) = \sum_{j=0}^{j=n-1} \alpha_{ij} t^{n-1-j}$$

from the n linear equations

$$\phi_i(t) = i_t \qquad (t = 1, 2, 3, \cdots, n).$$

The results are

$$\alpha_{ij} = \frac{\begin{vmatrix} 1^{n-1} & 1^{n-2} & \cdots & i_1 & 1^{n-2-j} & \cdots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & i_2 & 2^{n-2-j} & \cdots & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n^{n-1} & n^{n-2} & \cdots & i_n & n^{n-2-j} & \cdots & n & 1 \end{vmatrix}}{\pm\sqrt{\Delta}} \qquad (j = 0, 1, 2, 3, \cdots, n-1),$$

*The theorems used here are known as properties A and B of the Galois group. See DICKSON, *Introduction to the theory of algebraic equations*, p. 53.

where

$$\pm\sqrt{\Delta}=\begin{vmatrix} 1^{n-1} & 1^{n-2} & \cdots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & 2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ n^{n-1} & n^{n-2} & \cdots & n & 1 \end{vmatrix}=\prod_{r=1}^{r=n-1}(n-r)!.$$

§ 5. *Special examples.*

1. Let $n=3$ and

$$S_1=(x_1x_2x_3), \qquad S_2=(x_1x_2).$$

Then

$$f(t)=t^3-6t^2+11t-6, \qquad \phi_1(t)=-\tfrac{3}{2}t^2+\tfrac{11}{2}t-2,$$

$$\phi_2(t)=\tfrac{3}{2}t^2-\tfrac{11}{2}t+6.$$

These define the symmetric group on three letters.

2. Let $n=4$ and

$$S_1=(x_1x_2x_3x_4), \qquad S_2=(x_1x_2)(x_3x_4), \qquad S_3=(x_1x_2x_3).$$

Then

$$f(t)=t^4-10t^3+35t^2-50t+24, \qquad \phi_1(t)=-\tfrac{2}{3}t^3+4t^2-\tfrac{19}{3}t+5,$$

$$\phi_2(t)=-\tfrac{4}{3}t^3+10t^2-\tfrac{65}{3}t+15, \qquad \phi_3(t)=\tfrac{4}{3}t^3-\tfrac{19}{2}t^2+\tfrac{121}{6}t-10.$$

These define the symmetric group on four letters.

§ 6. *Rational fractional transformations.*

The results of the previous articles can be extended to rational fractional transformations.

Consider a group G of transformations

$$T_i\equiv[x:\psi_i(x)],$$

where

$$\psi_i(x)=\frac{\phi_i(x)}{\theta_i(x)}, \qquad \phi_i(x)=\sum_{j=0}^{j=m_i}\alpha_{ij}x^{m_i-j}, \qquad \theta_i(x)=\sum_{j=0}^{j=n_i}\beta_{ij}x^{n_i-j}.$$

$\alpha_{i0}\neq 0$, $\beta_{i0}\neq 0$, while $\phi_i(x)$ and $\theta_i(x)$ have no common factor and at least one of the degrees m_i, n_i exceeds unity.

The coefficients α_{ij} and β_{ij} are elements of a general field F and the quantity x belongs to a set X in a field F'. As before, these transformations are associative and have the closure property. If T_i and $T_{i'}$ are inverses

$$T_iT_{i'}\equiv[x:x]=[x:\psi_i\{\psi_{i'}(x)\}]$$

and we have

$$\psi_i\{\psi_{i'}(x)\}=x \qquad (m_in_i>1).$$

This is either (*a*) *an equation of condition*, $f(x) = 0$, or (*b*) *an identity*.

(*a*) In this case the transformations reduce $[\bmod f(x)]$ to the integral form considered in the first part of the paper.*

(*b*) In this case,

$$y = \frac{\phi_i(x)}{\theta_i(x)} \quad \text{gives} \quad x = \frac{\phi_{i'}(y)}{\theta_{i'}(y)},$$

therefore to each y there is only one x and therefore $\phi_i(x)$ and $\theta_i(x)$ are linear. Case (*b*) is therefore excluded.

§ 7. *Representation of products of substitutions.*

Consider any k substitutions R_j of order r_j $(j = 1, 2, \cdots, k)$ on the n roots of $f(x) = 0$.

Take the products of powers of these substitutions of the form †

$$S_i = R_1^{y_1^{(i)}} R_2^{y_2^{(i)}} \cdots R_k^{y_k^{(i)}} \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{bmatrix}.$$

The number of these products is

$$r = \prod_{j=1}^{j=k} r_j$$

and i will have the range $1, 2, \cdots, r$.

When the basic substitutions $R_j (j = 1, 2, \cdots, k)$ are given, S_i will be determined by the exponents $y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)}$.

It is possible to represent all these substitutions by the transformations

$$T_i \equiv [x : \phi(x; y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})] \tag{4}$$

where ϕ is determined by the generalized Lagrangian interpolation formula

$$\phi(x; y_1, y_2, \cdots, y_k) = \sum_{l=1}^{l=n} \sum_{j=1}^{j=r} \frac{x_{j_l} f(x)}{(x - x_l) f'(x_l)} \prod_{p=1}^{p=k} \frac{\theta_p(y_p)}{(y_p - y_p^{(j)}) \theta'_p(y_p^{(j)})}$$

and

$$f(x) = \prod_{t=1}^{t=n} (x - x_t), \qquad \theta_p(y_p) = \prod_{s=1}^{s=r} (y_p - y_p^{(s)}).$$

When any particular set of y's as $(y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})$ are substituted in the above it reduces to the regular Lagrangian formula and gives the $\phi_i(x)$ used in first part of this paper and therefore T_i. The function ϕ is a rational integral

* H. WEBER, *Lehrbuch der Algebra*, vol. 1, p. 170.

† No two sets $(y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})$ are alike but no assumption is made concerning the corresponding S_i.

function of x whose coefficients are rational integral functions of the k parameters $y_1, y_2, \cdots, y_k$. The numerical coefficients will be contained in the field F when S_i fulfills the conditions in Theorem II for every value of i.

Any set of substitutions S_i $(i = 1, 2, \cdots, r)$ where each substitution is characterized by a particular set of values $y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)}$ of the k parameters $y_1, y_2, \cdots, y_k$ can be represented by transformations T_i determined as above. *It is therefore possible to represent* an entire group of transformations by a single formula* (4).

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*Some of the transformations may be repeated.

ON OSCULATING ELEMENT-BANDS ASSOCIATED WITH LOCI OF SURFACE-ELEMENTS*

BY

PERCEY F. SMITH

In a memoir by SCHEFFERS, entitled *Isogonalcurven, Aequitangentialcurven und complexe Zahlen*, which appeared in volume 60 of the Mathematische Annalen (1905), p. 491, a number of theorems were demonstrated concerning properties of the osculating circles of certain loci of line-elements in the plane. Two of these may be cited here. Given a differential equation of the first order,

$$f(x, y, p) = 0, \tag{1}$$

in which $p = dy/dx$, and the one-parameter group

$$x = x, \qquad y = y, \qquad \text{arc tan } p = \text{arc tan } p' + t, \tag{2}$$

by which each line-element is turned about its point through the same angle, then the transformed elements satisfy the equation

$$f\left(x, y, \frac{p' + \tan \tau}{1 - p' \tan \tau}\right) = 0, \tag{3}$$

the integral curves of which are isogonal trajectories of the integral curves of the original equation (1). The first theorem in question is now this: The osculating circles of all the isogonal trajectories through a fixed point pass also through a second fixed point, that is, form a pencil. Furthermore, these circles will osculate at the second point another system of isogonal trajectories.

The second theorem is similar in character if the group (2) is replaced by

$$x = \bar{x} + \frac{t}{\sqrt{1 + p^2}}, \qquad y = \bar{y} + \frac{pt}{\sqrt{1 + p^2}}, \qquad p = p, \tag{4}$$

which has the effect of *sliding* each line-element along its line the same distance t. The integral curves of the transformed equation

$$f\left(\bar{x} + \frac{t}{\sqrt{1 + p^2}}, \quad \bar{y} + \frac{pt}{\sqrt{1 + p^2}}, \quad p\right) = 0 \tag{5}$$

are called by SCHEFFERS *aequitangential curves* of the original system (1). Then

* Presented to the Society, September 15, 1909, and February 26, 1910.

the theorem holds: The osculating circles of the aequitangential curves which touch a given line will touch also a second line. Furthermore, these circles will osculate on the second line a second system of aequitangential curves.

The discussion of the present paper leads to similar theorems for loci of surface-elements in space. Such loci are regarded as associations of *element-bands*, and either of two species of simple bands — parabolic and cubic — appear in the same role as osculating circles in the plane. The investigation is much simplified by means of transformations which make the problem depend upon projective geometry in space of five dimensions. This method has been employed to advantage by EIESLAND in a paper referred to below.

As far as the author is aware, the question of osculating element-bands is discussed for the first time in this paper.

§1. *Preliminary Theorems.*

Two united line-elements in the plane determine a unique curvature-element (x, y, y', y''). For, given two such elements (x, y, y') and $(x + dx, y + dy, y' + dy')$, where $dy = y'\,dx$, then if y'' is defined by the equation $dy' = y''\,dx$, it is clear that the ratios $dx : dy : dy'$ determine y'' uniquely.

The case is different, however, with two united surface-elements in space. For convenience we may adopt the notation (P, E) for such an element, P being the point and E the plane of the element. Assuming two united elements (P, E) and (P', E') whose coördinates are (x, y, z, p, q) and $(x + dx, y + dy, z + dz, p + dp, q + dq)$, where $dz = p\,dx + q\,dy$, then of the ratios

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{p\alpha + q\beta} = \frac{dp}{\delta} = \frac{dq}{\epsilon}, \tag{1}$$

three are independent. On introducing the numbers r, s and t defined by the equations

$$dp = r\,dx + s\,dy, \qquad dq = s\,dx + t\,dy, \tag{2}$$

or also, from (1),

$$\delta = r\alpha + s\beta, \qquad \epsilon = s\alpha + t\beta, \tag{3}$$

it is obvious that r, s, t are *not* determined uniquely when the ratios (1) are given, but that, in fact, we have the result: * *Two united surface-elements in space determine* ∞^1 *curvature-elements* (x, y, z, p, q, r, s, t).

Consider now any surface to which the united elements under discussion belong. The curvature of normal sections at P is, by the well-known formula, given by

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \cdot \frac{r\,dx^2 + 2s\,dx\,dy + t\,dy^2}{ds^2}, \tag{4}$$

* ENGEL, *Die höheren Differentialquotienten*, Leipziger Berichte (1893), p. 475.

where ds is the linear element. If we substitute r and t in terms of s from (3), then (4) becomes

$$\frac{1}{R} = \frac{1}{\sqrt{1+p^2+q^2}} \cdot \frac{\beta\delta dx^2 + \alpha\epsilon dy^2 - s(\beta dx - \alpha dy)^2}{\alpha\beta ds^2}, \tag{5}$$

in which of course, for any given value of $dx : dy$, s is arbitrary. We observe, however, that the curvature is the same for all values of s when the normal section is such that $\beta dx - \alpha dy = 0$, that is, such that (1) is satisfied. The result may be expressed as follows: *The ∞^1 curvature-elements determined by two united surface-elements (P, E) and (P', E') hang together along the direction PP' in such a manner that the curvature at P of all normal sections containing PP' is the same.*

From equations (3), we readily find that

$$\alpha\beta(rt - s^2) = -(\alpha\delta + \beta\epsilon)s + \delta\epsilon. \tag{6}$$

If, then, $\alpha\delta + \beta\epsilon \neq 0$, one of the curvature-elements has zero total curvature. On the other hand, when $\alpha\delta + \beta\epsilon = 0$, the total curvature of all the ∞^1 curvature-elements is the same.

Finally, denoting by $\delta x : \delta y$ the direction conjugate to $dx : dy$ on any surface containing the pair of elements under consideration, then the usual formula $rdx\delta x + s(dx\delta y + dy\delta x) + tdy\delta y = 0$ gives by substitution from (3), and on solving for $\delta x : \delta y$,

$$\frac{\delta x}{\delta y} = -\frac{\alpha\ \epsilon dy + s(\beta dx - \alpha dy)}{\beta\ \delta dx - s(\beta dx - \alpha dy)}. \tag{7}$$

Hence if $\beta dx - \alpha dy = 0$, the conjugate direction is the same for all values of s, namely, $\delta\delta x + \epsilon\delta y = 0$. In words: *On all surfaces containing two united surface-elements (P, E) and (P', E') the direction conjugate to PP' is the same.*

This fact is obvious geometrically. For this common conjugate direction is determined by the line of intersection of the planes E and E' of the united elements.

Evidently, if $\alpha\delta + \beta\epsilon = 0$, the conjugate direction is identical with PP', and the latter direction is therefore an asymptotic direction. This fact when combined with the conclusion found in discussing (6) gives the result: *If the total curvature of the ∞^1 curvature-elements (3) is the same, the direction along which they are united is an asymptotic direction.*

§ 2. *Parabolic and Cubic Bands.*

In the Differential Geometry of the plane, the circle plays a conspicuous rôle. This fact, when metric properties are left out of consideration, is explained by remarking that a circle is uniquely determined by two of its united elements. Plainly, a curve of any family depending upon three parameters will have this

last property — vertical parabolas, for example. The point is that we consider *unions of line-elements which are determined when two united elements are given.*

An analogous problem for space now arises. We wish to consider unions of ∞^1 surface-elements each one of which is determined if two united elements are given. Let the term *element-band* be used for a union of ∞^1 surface-elements. Analytically an element-band is defined by

$$(1) \quad x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad p = p(t), \qquad q = q(t),$$

provided that $z' = px' + qy'$, where accents indicate derivatives with respect to the parameter t.

Two species of element-bands, now to be described, appear in this paper. The points of the elements lie on a curve, which may be called the *point-locus*, while the planes form a developable, called the *plane-locus*. The two species of element-bands may be respectively named *parabolic bands* and *cubic bands*.

The point-locus of a parabolic band is a parabola whose axis is parallel to the Z-axis. The plane-locus is a parabolic cylinder. To conceive such a band we may think of such a parabola, which may be called a *vertical parabola* then imagine a line meeting the parabola to generate a cylinder, and finally picture to ourselves a narrow band of the cylindrical surface along the parabola. Clearly, ∞^2 parabolic bands may have a common point-locus or dually, the same plane-locus. To dwell for a moment on the last point: — the parabolic cylinders in question obviously contain the point at infinity on the Z-axis. Given such a cylinder, then any plane through the point referred to determines on the cylinder a parabolic band.

The point-locus of a cubic band is a skew cubic. The plane-locus is a quadric cone whose vertex lies on the cubic. Clearly there are ∞^1 cubic bands with a common point-locus. On the other hand, the multiplicity of the system of cubic bands belonging to a quadric cone is the same as that of the system of skew cubics lying on it, and is of no special importance at this moment. The concept of a cubic band is readily visualized if we think of a skew cubic, picture any quadric cone standing on it and think of a narrow band of the conical surface along the cubic.

The fact is now readily established that a parabolic band is determined by two of its united elements, (P, E) and (P', E'). For pass through PP' the plane parallel to the Z-axis. The united line-elements in this plane determined by its intersection with E and E' determine a vertical parabola. Further, the line of intersection of E and E' gives an element of the parabolic cylinder. The multiplicity of all parabolic bands is obviously seven-fold. For there are ∞^8 pairs of united elements and ∞^1 elements determine the same parabolic band. The important exceptional case already noted — when E and E' intersect along PP' — leads to a *twisted band*. Namely, the point-locus is now the

line PP' and the planes of the plane-locus pass through PP'. The points and planes of the elements of the band are in a projective relation, such that P and E, P' and E' are corresponding elements, and moreover, the point at infinity on PP' corresponds to the vertical plane through PP'.

Turn next to the question of the determination of a cubic band by a pair of united surface-elements. From the results of § 1, it is clear that the vertex of any cone to which the united elements (P, E) and (P', E') belong, must lie on the line of intersection of E and E'. For this direction is conjugate to PP', and for the present we assume these directions are not identical. Denote the line of intersection of E and E' by (E, E'). Let us now confine ourselves to quadric cones. Consider the normal section of such a cone through the direction PP'. Then by § 1, the curvature of this conic at P is given. Consequently three conditions are imposed upon this conic. It is therefore clear that ∞^3 quadric cones may contain the same pair of united elements. The cones touch along a common element (E, E') and in addition, their normal sections through PP' osculate at P. To obtain a unique cone of this system we may introduce the three fixed conditions — *the cones shall pass through three non-collinear fixed points* A, B *and* C. It is easy to see that the required cone is now uniquely determined. For, in the first place, its trace c' on the plane of A, B and C is determined. Secondly, the normal section through PP' now has *two* points given on this trace c', and hence is a unique conic c''. Only one cone with a *given* element can contain two conics.

We see, therefore, that a pair of united surface-elements (P, E), (P', E') will determine a unique quadric cone K standing on three fixed points A, B, C. Let the vertex of this cone be V. Consider the quadric cone whose vertex is A which stands on the five points P, P', V, B and C. Its intersection with the first cone is a skew cubic C_3 and the element VA. Similarly each of the fixed points B and C is the vertex of a quadric cone which will contain the *same* cubic. The cubic band lying on K along C_3 contains (P, E) and (P', E') and furthermore is uniquely determined by them.

The multiplicity of skew cubics passing through three non-collinear fixed points is six-fold. There are accordingly ∞^7 cubic bands, ∞^1 on each of the skew cubics of the system. On the other hand, on each quadric cone standing on the fixed points A, B and C lie ∞^2 cubic bands, and the multiplicity of such cones is five-fold.

If the line of intersection of the planes E and E' is identical with PP', the cubic band is constructed by assuming E as the osculating plane at P, and PP', of course, as the tangent. The cubic satisfying this condition and moreover passing through A, B, and C is unique. The cone K then has its vertex at P.

§ 3. *Osculating Bands.*

Two element-bands having in common two united elements may be said to *osculate*. The analytical conditions are easily derived. For given two bands

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad p = \phi(t), \quad q = \chi(t),$$

and

$$x = f_1(t), \quad y = g_1(t), \quad z = h_1(t), \quad p = \phi_1(t), \quad q = \chi_1(t),$$

then osculation demands the equality of f and f_1, g and g_1, etc., and their first derivatives for some value of t. In these conditions, however, the relation $h' = h_1'$ depends upon the others from the conditions for bands, (1), § 2.

Geometrically, two element-bands osculate when their point-loci touch and their plane-loci are tangent along a common generator.

The immediate subject of investigation concerns properties of surface-element loci with respect to osculating parabolic or cubic bands. It is clear that there are ∞^1 bands of this character associated with any element-band. Further, given any surface or union of ∞^2 elements, then ∞^3 osculating parabolic or cubic bands are determined. Of particular interest, however, is the case when the surface-element locus is defined by a partial differential equation of the first order

$$f(x, y, z, p, q) = 0. \tag{1}$$

The ∞^4 elements satisfying (1) are arranged on ∞^3 bands, the characteristic bands. If the coördinates of two united elements of a characteristic band are (x, y, z, p, q) and $(x + dx, y + dy, z + dz, p + dp, q + dq)$, then these coördinates of course satisfy $f = 0$ and further the well-known equations due to Lie,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}. \tag{2}$$

Each element satisfying (1) is associated with a united element determined by (2) such that the pair of united elements lie on a characteristic band. Now, clearly, the parabolic or cubic band determined by this pair of elements will osculate the characteristic band. Thus at each element of the locus (1), there is determined a parabolic or cubic band which osculates the characteristic band containing the given element. There are ∞^4 osculating parabolic or cubic bands. If, now, we look upon the locus of (1) not as a set of integral surfaces, but as an association of ∞^3 element-bands, the characteristic bands, then the properties of osculating parabolic or cubic bands appear a natural subject for investigation. The ∞^4 parabolic or cubic bands which osculate the characteristic bands of a given differential equation of the first order we shall refer to briefly as the osculating parabolic or cubic bands of the locus of the equation.

§ 4. *Transformation of the Problem.*

The investigation of the question proposed in the preceding section is greatly facilitated by the aid of transformations by which the geometry of surface-elements is made to depend upon projective geometry in a space of five dimensions R_5. Such a transformation has been given by LIE.* A second transformation is exhibited in § 7 of this paper. The capital point is that parabolic or cubic bands correspond by these respective transformations to straight lines in R_5. To disclose the details, we begin by examining the transformation already referred to as due to Lie. This is defined by the equations

$$(1)\quad X_1 = x, \qquad X_2 = y, \qquad Y_1 = \tfrac{1}{2}p, \qquad Y_2 = \tfrac{1}{2}q, \qquad Z = z - \tfrac{1}{2}px - \tfrac{1}{2}qy,$$

by which identically

$$(2)\qquad dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2 = dz - pdx - qdy.$$

If we interpret (X_1, X_2, Y_1, Y_2, Z) as cartesian point coördinates in R_5, then clearly united surface-elements in ordinary space R_3 transform into *line-elements* in R_5 satisfying the equation

$$(3)\qquad dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2 = 0.$$

Now equation (3) is the differential equation of the *null-system*,†

$$(4)\qquad Z - Z' + Y_1 X_1' - X_1 Y_1' + Y_2 X_2' - X_2 Y_2' = 0,$$

that is, of a correlation in R_5 such that point and corresponding linear M_4 are *incident*.

A line in R_5,

$$(5)\quad X_1 = X_1^0 + \lambda_1 t, \quad X_2 = X_2^0 + \lambda_2 t, \quad Y_1 = Y_1^0 + \mu_1 t, \quad Y_2 = Y_2^0 + \mu_2 t, \quad Z = Z^0 + \nu t,$$

whose elements satisfy (3), that is, such that

$$(5')\qquad \nu + \mu_1 X_1 - \lambda_1 Y_1 + \mu_2 X_2 - \lambda_2 Y_2 = 0,$$

is a *null-line*. There are plainly ∞^7 null-lines.

By this transformation (1), the null-line in R_5 becomes in R_3 an element-band. This is a parabolic band, as shall now appear. Transforming (5), we obtain for the equations of the corresponding band, using (5′),

$$(6)\quad \begin{gathered} x = x^0 + \lambda_1 t, \qquad y = y^0 + \lambda_2 t, \qquad z = z^0 + (\lambda_1 p^0 + \lambda_2 q^0)t + (\lambda_1\mu_1 + \lambda_2\mu_2)t^2, \\ p = p^0 + 2\mu_1 t, \qquad q = q^0 + 2\mu_2 t. \end{gathered}$$

The point-locus of this band is, in the general case, a vertical parabola.‡ The

* LIE-ENGEL, *Transformationsgruppen*, vol. 2, p. 521.

† For details, see EIESLAND, *On Null-Systems in Space of Five Dimensions*, etc., American Journal of Mathematics, vol. 26 (1904), p. 103.

‡ EIESLAND, l. c., p. 120.

plane-locus is the envelope of

$$z'-z=p(x'-x)+q(y'-y), \tag{7}$$

in which x, y, z, p, q, have the values in (6). On substituting their values in (7), the equation takes the form

$$(\lambda_1\mu_1+\lambda_2\mu_2)t^2-2[\mu_1(x'-x^0)+\mu_2(y'-y^0)]t+z'-z^0-p^0(x'-x^0)-q^0(y'-y^0)=0. \tag{8}$$

The envelope of (8) is clearly, in the general case, a parabolic cylinder. The band (6) is accordingly a parabolic band. Certain exceptional cases are necessary for the following discussion.

1. If $\lambda_1=\lambda_2=0$, then (6) becomes $x=x^0$, $y=y^0$, $z=z^0$. That is, the point-locus is a point. The ∞^1 surface-elements now form a *hinge*, for the planes (8) form a pencil.

2. If $\lambda_1\mu_1+\lambda_2\mu_2=0$ and $\lambda_1\neq 0$, $\lambda_2\neq 0$, the point-locus is a line and the plane-locus a plane-pencil. The band is now a *twisted band*, the points and planes of the element forming a projective series.

3. If $\mu_1=\mu_2=0$, the band just discussed becomes *flat*, that is, a straight band on a plane.

It is clear from these results that the contact geometry of parabolic bands in R_3 becomes under the transformation (1) the *projective* geometry of a null-system. The group of the latter is easily found to depend upon 21 parameters.*

In later sections it will be shown that the null-lines (5) are by another transformation changed into cubic bands, and finally that parabolic and cubic bands correspond under a contact transformation.

§ 5. *Theorems on the Null-System in R_5.*

Certain theorems regarding the null-system

$$Z-Z'+Y_1X_1'-X_1Y_1'+Y_2X_2'-X_2Y_2'=0, \tag{1}$$

necessary in the sequel, are discussed in this section.

The plane in R_5

$$Y_1=a_1X_1+a_2X_2+a_3, \qquad Y_2=b_1X_1+b_2X_2+b_3, \qquad Z=c_1X_1+c_2X_2+c_3 \tag{2}$$

will contain the null-line (5), § 4, if in addition to the obvious conditions that the point (X_1^0, X_2^0, etc.) lies on (2), we have

$$\mu_1=a_1\lambda_1+a_2\lambda_2, \qquad \mu_2=b_1\lambda_1+b_2\lambda_2, \qquad \nu=c_1\lambda_1+c_2\lambda_2. \tag{3}$$

On substituting in the condition (5′), § 4, for a null-line, the single condition is obtained

$$[c_1-a_3+(b_1-a_2)X_2^0]\lambda_1+[c_2-b_3+(a_2-b_1)X_1^0]\lambda_2=0. \tag{4}$$

*Eiesland, l. c., p. 142.

The following conclusions are now apparent:

1. Through an arbitrary point in an arbitrary plane in R_5 there is one null-line.

2. *All* lines in the plane (2) through the point determined by

$$X_1^0 = \frac{b_3 - c_2}{a_2 - b_1}, \qquad X_2^0 = \frac{c_1 - a_3}{a_2 - b_1}, \tag{5}$$

are null-lines. This point is the *null-point* of the plane.

3. If $a_2 = b_1$, $b_3 = c_2$, $c_1 = a_3$, then *every* line in the plane is a null-line. The plane (2) is now a *null-plane*. The conditions obviously demand *symmetry* in the determinant of the right-hand members in (2). Explicitly, the equations (2) take the form, if the plane is a null-plane,

$$Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = a_2 X_1 + b_2 X_2 + c_2, \qquad Z = a_3 X_1 + c_2 X_2 + c_3. \tag{6}$$

If we next inquire how many null-planes contain a given null-line, the answer is immediate, namely, *each null-line is the axis of a pencil of* ∞^1 *null-planes.*

Clearly, all planes in R_5 cannot be represented in the form (2). The two exceptional cases are, for null-planes, given by

$$X_1 = a, \qquad X_2 = b, \qquad Z = c_3 - a Y_1 - b Y_2, \tag{7}$$

and

$$X_1 = a X_2 + b, \qquad Y_2 = c X_2 - a Y_1 + e, \qquad Z = e X_2 - b Y_1 + h. \tag{8}$$

We may, in general, define a null-surface* in R_5 as one whose tangent planes are null-planes.

Turning now to the locus of *one* equation

$$F(X_1, X_2, Y_1, Y_2, Z) = 0, \tag{9}$$

defining in R_5 a manifoldness M_4 of four dimensions, the linear *tangent* M_4^1 is given as usual by

$$\begin{aligned} F_{X_1}(X_1' - X_1) + F_{X_2}(X_2' - X_2) + F_{Y_1}(Y_1' - Y_1) \\ + F_{Y_2}(Y_2' - Y_2) + F_Z(Z - Z) = 0. \end{aligned} \tag{10}$$

The null-plane (6) will pass through the point of contact (X_1, X_2, etc.) and lie entirely within the locus of (10) under these conditions:

$$a_3 = Y_1 - a_1 X_1 - a_2 X_2, \qquad c_2 = Y_2 - a_2 X_1 - b_2 X_2, \qquad c_3 = Z - a_3 X_1 - c_2 X_2, \tag{11}$$

$$F_{X_1} + a_1 F_{Y_1} + a_2 F_{Y_2} + a_3 F_Z = 0, \qquad F_{X_2} + a_2 F_{Y_1} + b_2 F_{Y_2} + c_2 F_Z = 0. \tag{12}$$

There is no difficulty in showing that the ∞^1 null-planes satisfying (11) and (12) form a *pencil* whose axis is the null-line for which

$$\frac{dX_1}{F_{Y_1} - X_1 F_Z} = \frac{dX_2}{F_{Y_2} - X_2 F_Z} = \frac{dY_1}{-F_{X_1} - Y_1 F_Z} = \frac{dY_2}{-F_{Y_2} - Y_2 F_Z} = \frac{dZ}{\Sigma X F_X}, \tag{13}$$

* For a development of this idea, see EIESLAND, l. c., p. 128.

the denominator ΣXF_X containing the four terms corresponding to X_1, X_2, Y_1, Y_2. We may express this important result as follows:

At each point of an M_4 in R_5 there is a pencil of tangent null-planes whose axis is the null-line (13).

The null-M_4^1 of P contains every null-plane through P. Hence the result just found may be formulated thus:

The tangent M_4^1 and null-M_4^1 of P intersect in a pencil of null-planes.

We shall call the axis of this pencil the *tangent null-axis at P*.

The result just stated is fundamental for the sequel. It is clear that the tangent null-axis at any point on the locus of (9) is distinguished from all other tangent null-lines, of which there are of course ∞^2.

Returning to the plane (2), we may represent this for the moment by the symbol (abc) of the determinant on the right-hand members. As already pointed out, if (abc) is symmetrical, the plane is a null-plane. In the null-system (1), let the point (X_1, X_2, etc.) describe the plane (abc). Then it appears at once that the ∞^2 corresponding M_4^1's intersect in a plane *whose symbol is the conjugate of* (abc). Such planes we may call *conjugate planes* in the null-system. Clearly, a null-plane is *self-conjugate.*

In the null-system (4), to every point corresponds a linear M_4. Dually, to a given linear M_4 will correspond a point, the null-point of the M_4. This fact leads to the determination of the locus of the null-point of the linear tangent M_4 given by (10). To find this null-point $P_0(X_1^0, X_2^0, Y_1^0, Y_2^0, Z^0)$, we merely have to compare equation (10) with the equation of the null-system

$$Z - Z^0 + Y_1X_1^0 - X_1Y_1^0 + Y_2X_2^0 - X_2Y_2^0 = 0. \tag{14}$$

The result is readily found to be

$$X_1^0 = \frac{F_{Y_1}}{F_Z}, \qquad X_2^0 = \frac{F_{Y_2}}{F_Z}, \qquad Y_1^0 = -\frac{F_{X_1}}{F_Z}, \qquad Y_2^0 = -\frac{F_{X_2}}{F_Z},$$
$$Z^0 = Z + \frac{X_1F_{X_1} + X_2F_{X_2} + Y_1F_{Y_1} + Y_2F_{Y_2}}{F_Z}. \tag{15}$$

When the point $P(X_1, X_2, Y_1, Y_2, Z)$ describes the locus of (9), the point P^0, which we call the *conjugate point*, will describe a locus, which may be named the *conjugate locus.* In the general case, the conjugate locus will be of four dimensions, say $F^0 = 0$. Evidently, from the definition of the conjugate locus, it is derived from the original locus by the correlation established in R_5 by the null-system. In this correlation each null-plane is invariant. Hence the pencil of null-planes tangent to $F = 0$ at P must have the same relation to the conjugate locus at P^0. In other words, the relation between two conjugate loci is *mutual.* The linear tangent M_4 and the null-M_4^1 at P to $F = 0$ are respectively

the null-M_4^1 and linear tangent M_4 to $F^0 = 0$ at P^0. From the theorem on tangent null-planes at P, we now derive the important

THEOREM. *If the loci of $F = 0$ and $F^0 = 0$ are conjugate under a null-system in R_5, and if P and P^0 are corresponding points on these loci, then the ∞^1 null-planes tangent to $F = 0$ at P are tangent also to $F^0 = 0$ at P^0. Further, the tangent null-axes at P and P^0 are identical.*

Analytically, the general case arises when the tangent M_4^1 (10) contains *four* independent parameters. The null-point P_0 will then describe a four dimensional locus. If, then, the number of independent parameters in (10) is less than four, the locus of P_0 will be of three dimensions, or of two (a surface) or of one (a curve). All these cases will be covered if we consider the "envelope" of the linear M_4,

$$E \equiv A_1 X_1 + A_2 X_2 + B_1 Y_1 + B_2 Y_2 + CZ = 0, \tag{17}$$

in which the coefficients are functions of certain parameters in number less than four. For example, consider the case of two parameters u and v. Taking the first partial derivatives of (17) with respect to these parameters,

$$E_u = 0 \quad \text{and} \quad E_v = 0, \tag{18}$$

we see that the tangent null-axes at all the points in the plane $E = E_u = E_v = 0$ pass through a common point, the null-point of $E = 0$. Thus the ∞^4 tangent null-axes form in this case ∞^2 bundles. The two other special cases mentioned are clearly those in which the tangent null-axes form ∞^3 flat pencils or ∞^1 three-dimensional *fascicules*. In the former of these two cases, that of three parameters, the points of tangency of the axes of a pencil lie on a line.

§ 6. *Geometry of Osculating Parabolic Bands.*

The results found in the previous section are now to be interpreted in space of three dimensions by means of Lie's transformation used in § 4,

$$X_1 = x, \quad X_2 = y, \quad Y_1 = \tfrac{1}{2}p, \quad Y_2 = \tfrac{1}{2}q, \quad Z = z - \tfrac{1}{2}px - \tfrac{1}{2}qy. \tag{1}$$

We obtain first some preliminary theorems. To *any* line in R_5 not a null-line, will correspond in R_3 a surface-element locus, for which

(a) the point-locus is a vertical parabola,

(b) the plane-locus is a parabolic cylinder.

These results follow directly as in § 4, the distinction being that the condition for a null-line simply implies that the point-locus is on the plane-locus.

To any plane in R_5 [(2), § 5] will correspond in R_3 a locus of ∞^2 elements, with the properties

(a) the point-locus is a vertical paraboloid,

(b) the plane-locus is a second vertical paraboloid.

In fact, the plane

(2) $Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = b_1 X_1 + b_2 X_2 + b_3, \qquad Z = c_1 X_1 + c_2 X_2 + c_3,$

gives under (1) as the point-locus,

(3) $$z = a_1 x^2 + (a_2 + b_1) xy + b_2 y^2 + (a_3 + c_1) x + (b_3 + c_2) y + c_3.$$

The statement (b) leads to a simple envelope problem.

The paraboloid (3) is the point-locus of ∞^3 assemblages of ∞^2 surface elements, namely, for all such that $a_2 + b_1$, $a_3 + c_1$, and $b_3 + c_2$ are constant. One and only one of these assemblages is a union, namely, when $a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$, and in this case, the union consists of the elements of the paraboloid. Remarks of like character apply to the plane-locus.

The theorems 1 and 2 on page 309 enable us to state these properties of the point-locus or the plane-locus: *The ∞^2 surface-elements are arranged in ∞^1 parabolic bands having a common element which belongs to both point-locus and plane-locus. The point-locus and the plane-locus touch on this element. Two conjugate planes in R_5 have the same point-loci and the same plane-loci.*

To the *null-plane*

(4) $Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = a_2 X_1 + b_2 X_2 + c_2, \qquad Z = a_3 X_1 + c_2 X_2 + c_3$

correspond the ∞^2 surface-elements of the paraboloid

(5) $$z = a_1 x^2 + 2a_2 xy + b_2 y^2 + 2a_3 x + 2c_2 y + c_3.$$

If $a_1 b_2 = a_2^2$, the locus of (5) is a cylinder. That is, the ∞^2 parabolic bands of the locus have now the *same plane-locus*. Obviously, equation (5) gives also all planes as a special case.

The special null-planes (7) and (8) of § 6 yield in R_3 the following results. The null-plane

(6) $$X_1 = a, \qquad X_2 = b, \qquad Z = aX_1 + bX_2 + c,$$

gives the ∞^2 elements of the point (a, b, c). To the null-plane

(7) $X_1 = aX_2 + b, \qquad Y_2 = cX_2 - aY_1 + e, \qquad Z = eX_2 - bY_1 + h,$

correspond the ∞^2 elements of the vertical parabola *

(8) $$x = ay + b, \qquad z = cy^2 + 2ey + h.$$

The discussion shows that the ∞^2 surface elements of a point or a plane, or of a vertical parabola, or of a parabolic cylinder, correspond to the points of a plane in R_5.

In the previous section it has been shown that a null-line is the axis of a pencil of null-planes. The corresponding result for space of three dimensions is this:

* EIESLAND, l. c., p. 121.

A parabolic band lies on ∞^1 vertical paraboloids.

The connection of these facts with the discussion of § 1 is simple. The point (X_1, X_2, etc.) in R_5 gives a surface-element in R_3. A consecutive point on a null-line in R_3 gives a united element. Through the point-pair in R_5 pass ∞^1 null-planes. The point (X_1, X_2, etc.) and each null-plane determine a curvature-element in R_5. For clearly, in addition to x, y, z, p, q determined by (1), we now have given in (5), $r = \frac{1}{2}a_1$, $s = \frac{1}{2}a_2$, $t = \frac{1}{2}b_2$. Thus the ∞^1 null-surface-elements in R_5 having a common null-axis correspond to ∞^1 curvature-elements with a common pair of united surface-elements.

The null-plane in R_5 transforms into an element union in R_3. More generally, any null-surface will transform into an element union — surface or curve.

The locus of

$$F(X_1, X_2, Y_1, Y_2, Z) = 0 \tag{9}$$

transforms into the partial differential equation of the first order

$$f(x, y, z, p, q) = 0. \tag{10}$$

The partial derivatives of (9) transform as follows:

$$\begin{gathered} F_{X_1} = f_x + \tfrac{1}{2}pf_z, \qquad F_{X_2} = f_y + \tfrac{1}{2}qf_z, \qquad F_Z = f_z, \\ F_{Y_1} = 2f_p + xf_z, \qquad F_{Y_2} = 2f_q + yf_z. \end{gathered} \tag{11}$$

Any null-surface lying entirely in (9) transforms into an integral surface of (10).

Now consider the pencil of tangent null-planes to (9) at any point, and the corresponding null-axis. Each of the ∞^1 corresponding vertical paraboloids will have in common with (10) a curvature-element, and the ∞^1 curvature elements thus arising will hang together on two united surface-elements. Hence *the null-axis transforms into a parabolic band which osculates a characteristic band.* This result is verified by transformation of the equations (13), § 5. For by the equations of the transformation (1) and (11) we obtain immediately,

$$\frac{dx}{2f_p} = \frac{dy}{2f_q} = \frac{\frac{1}{2}dp}{-f_x - pf_z} = \frac{\frac{1}{2}dq}{-f_y - qf_z} = \frac{dz}{2(pf_p + qf_q)},$$

that is, the equations (2) of § 3.

We may at once write down the equations of the parabolic band osculating the characteristic band of (10) at (x, y, z, p, q). They are ((6), § 4), since $\lambda_1, \lambda_2, \mu_1, \mu_2$ now satisfy (13) of § 5,

$$\begin{aligned} x' &= x + 2f_p t, & y' &= y + 2f_q t, \\ p' &= p - 2(f_x + pf_z)t, & q' &= q - 2(f_y + qf_z)t, \\ z &= z + 2(pf_p + qf_q)t - 2[f_x f_p + f_y f_q + f_z(pf_p + qf_q)]t^2. \end{aligned} \tag{12}$$

The exceptional cases noted on page 308 lead to the following results.

1. If $f_p = f_q = 0$, the osculating parabolic bands are *hinges*. The differential equation now contains no partial derivatives and its locus consists of all surface-elements of the points of the surface $f(x, y, z) = 0$.

2. If $f_x f_p + f_y f_q + f_z(pf_p + qf_q) = 0$ by virtue of $f = 0$, the osculating bands are twisted bands. The characteristic curves are now asymptotic lines on all integral surfaces of $f(x, y, z, p, q) = 0$. In fact, as explained in § 1, the direction PP' of the united elements of the band is now an asymptotic direction.

3. If $f_x + pf_z = f_y + qf_z = 0$, that is, if the differential equation is of the form $f(z - px - qy, p, q) = 0$, the osculating bands are flat.

Returning to 2, we see that the fact that the osculating bands are twisted states a *necessary and sufficient condition* that the characteristic curves shall be asymptotic lines on all integral surfaces. In this case, Lie has shown that the point-loci of the ∞^4 bands reduce to ∞^3 lines, that is, ∞^1 bands have the *same* point-loci. The general question may be asked:— *when do the* ∞^4 *osculating parabolic bands arrange themselves in* ∞^3 *systems of* ∞^1 *bands having a common point-locus?* This general problem includes the case just discussed.

If we write the equations (12) in the form

$$x = ay + b, \qquad z = cy^2 + 2ey + h,$$

then

$$(13) \qquad a \equiv f_p \div f_q, \qquad b = x - ay, \qquad c = -\tfrac{1}{2}\frac{f_x f_p + f_y f_q + f_z(pf_q + qf_q)}{f_q^2},$$

$$e = \tfrac{1}{2}(pa + q), \qquad h = z - cy^2 - 2ey.$$

The ∞^4 elements satisfying $f = 0$ lead by these formulas to the point-loci of the osculating parabolic bands. The analytic question is, therefore, when does the elimination of x, y, z, p, q from (13) and $f = 0$ lead to more than one equation in the a, b, c, e, h? Without answering this question in all generality, it is not difficult to show that any one of the following is a *sufficient* condition:—

1. Either a or c constant;

2. A functional relation between a and c.

An interesting case under 2 arises when

$$(13a) \qquad c = k(1 + a^2).$$

The ∞^4 osculating parabolic bands now lie on ∞^3 *congruent* vertical parabolas. The characteristic curves on all integral surfaces have the property expressed by the equation

$$(13b) \qquad dpdx + dqdy = k(dx^2 + dy^2).$$

Conversely, the condition (13b) is sufficient for the property of the osculating vertical bands (13a). Evidently $k = 0$ gives the case 2.

The equations (15) of the previous section become in space of three dimensions the transformation

$$x' = x + 2\frac{f_p}{f_z}, \qquad y' = y + 2\frac{f_q}{f_z}, \qquad p' = -p - 2\frac{f_x}{f_z},$$
$$q' = -q - 2\frac{f_y}{f_z}, \qquad z' = z - 2\frac{f_p f_x + f_q f_y}{f_z^2}. \tag{14}$$

Given any element of the locus of $f = 0$, equations (14) define an element (x', y', z', p', q'), which we call the *conjugate element*. Elimination of x, y, z, p and q from $f = 0$ by means of (14) will lead to the locus of the conjugate element—called the *conjugate locus*. In the general case, this elimination will give a single equation

$$f'(x', y', z', p', q') = 0. \tag{15}$$

The theorem of the preceding section now gives the interesting result.

THEOREM. *Given a partial differential equation of the first order. The* ∞^4 *parabolic bands which osculate the characteristic bands will in general osculate also the characteristic bands of a second partial differential equation of the first order which may be found by differentiation and elimination.*

The geometric relation between conjugate equations is simply this: The elements are grouped in conjugate pairs which belong to a parabolic band osculating the characteristic bands of both equations. The given equation may, of course, be *self-conjugate,* that is, $f = 0$ and $f' = 0$ may be identical. The equation $ff' = 0$, where $f = 0$ and $f' = 0$ are conjugate, offers a case of this kind.

If the characteristic curves are asymptotic lines on all integral surfaces of $f = 0$, that is, if the osculating parabolic bands are *twisted bands,* then the characteristics of $f' = 0$ have the same property. More generally, if the property (15) is possessed by the characteristics of $f = 0$, it holds also for those of $f' = 0$.

Referring to the equations of the osculating band (12) we see that the conjugate element (14) is obtained by setting the parameter t equal to $1 \div f_z$.

The preceding discussion assumes that $f_z \neq 0$, that is, that the given equation (10) contains z. If this is not the case and if one of the other variables x or y appears in the equation, say x, we merely need to write in (10) $p = 1 \div \partial x/\partial z$, $q = -\partial x/\partial y \div \partial x/\partial z$, and then interchange x and z. The case when neither x nor y nor z appears is excluded.

The exceptions to the theorem, p. 311, are readily characterized by the aid of the results given at the close of the preceding section. Transforming (16) of that section by (1) gives an equation which may be written

$$A_1 x + A_2 y + B_1 p + B_2 q + C(z - \tfrac{1}{2}px - \tfrac{1}{2}qy) + D = 0. \tag{16}$$

in which the coefficients A_1, A_2, etc., are functions of one, two or three parameters. The exceptional differential equations result, then, by forming first partial derivatives of (16) with respect to the parameters and then eliminating these, using (16). The following properties of the osculating parabolic bands characterize these differential equations.

Three parameters. The ∞^4 parabolic bands osculating the characteristic bands form ∞^3 families of ∞^1 each. The ∞^1 bands of a family osculate the characteristic bands at points lying on a vertical parabola and have a common surface-element. The latter, ∞^3 in all, constitute the conjugate locus.

Two parameters. The ∞^4 osculating parabolic bands now form ∞^2 families of ∞^2 each of this sort: the ∞^2 bands of a family have a common surface-element and osculate the characteristic bands at points lying on a vertical paraboloid. The conjugate locus consists of the ∞^2 common surface-elements of each family.

One parameter. The ∞^4 osculating parabolic bands are arranged in ∞^1 families of ∞^3 each such that all of one family have a common surface-element. The conjugate locus is made up of these latter elements, ∞^1 in all.

The facts here assembled follow at once from the conclusions at the end of § 5 and the results established in the present section.

An interesting theorem in regard to any locus of ∞^2 surface-elements

$$x = x(t, v), \qquad y = y(t, v), \qquad \cdots, \qquad q = q(t, v) \tag{17}$$

is obtained by the following considerations. The corresponding locus in R_5 is a surface S. Consider the tangent plane to this surface S at a point P. Through P and the null-point [(5), § 5] P' of the tangent plane passes a null-line. Thus at each point of S we determine a tangent null-line, and hence on S a system of ∞^1 *null-curves*. Each null-curve corresponds to an element band satisfying (17). Hence the

THEOREM. *Every locus of ∞^2 surface-elements in space may be arranged in ∞^1 bands.*

The locus of the null-point P' is a surface S' (if S is not a developable). Moreover, it is easily seen that the osculating plane of the null-curve at P is the tangent plane to S' at P'. Further P is the null-point of this plane. Hence the relation between S and S' is mutual. *The ∞^2 null-lines touching the null-curves on S touch also the null-curves on S'.* This result gives for space the

THEOREM. *Given an assemblage A of ∞^2 surface-elements in space. The ∞^1 bands in which these may be arranged determine ∞^2 osculating parabolic bands which in general osculate also the ∞^1 bands of a second assemblage A' of ∞^2 surface-elements.*

It appears, then, that assemblages of ∞^2 surface-elements in space are arranged in *conjugate pairs*, as the relation of A and A' may be named. If A is a union, then A' is identical with A.

The equations of A' are given directly by (17) when we substitute from (17) in the right-hand members. An exception to the theorem arises when S is a developable. The osculating parabolic bands are now arranged in ∞^1 families of ∞^1 each, the bands of a family having a common element and osculating the bands of the assemblage A in points lying on a vertical parabola. The assemblage A' consists of a locus of ∞^1 elements.

In preparation for certain final theorems in this section we now change the surface-element coördinates as follows. Write the equation of the plane of the element thus:

$$Z = uX + vY - w, \tag{18}$$

and take for the new coördinates of (P, E), u, v, w, α, β, the latter satisfying $dw - \alpha du - \beta dv = 0$. Then we readily find the relations

$$u = p, \qquad v = q, \qquad w = z - px - qy, \qquad \alpha = -x, \qquad \beta = -y, \tag{19}$$

between the old and new element-coördinates. The equations (12) of the parabolic band now become, if we assume that by (19),

$$f(x, y, z, p, q) = g(u, v, w, \alpha, \beta), \tag{20}$$

as follows:

$$\begin{aligned} u' &= u + 2g_\alpha t, & v' &= v + 2g_\beta t, \\ \alpha' &= \alpha - 2(g_u + \alpha g_w)t, & \beta' &= \beta - 2(g_v + \beta g_w)t, \\ w' &= w + 2(\alpha g_\alpha + \beta g_\beta)t - 2[g_u g_\alpha + g_v g_\beta + g_w(\alpha g_\alpha + \beta g_\beta)]t^2. \end{aligned} \tag{21}$$

The equations of the osculating parabolic band have, therefore, precisely the same form in the new coördinates. Furthermore, the same remark applies to equations (17) giving the coördinates of the conjugate element.

Let us now turn our attention to the question of *trajectory-bands* of a given differential equation

$$f(x, y, z, p, q) = 0. \tag{22}$$

Given a two-parameter group

$$x = x, \qquad y = y, \qquad z = z, \qquad p = \phi(x, y, z, \bar{p}, \bar{q}; t_1, t_2),$$
$$q = \chi(x, y, z, \bar{p}, \bar{q}; t_1, t_2),$$

by which the element (P, E) is turned about its point, then substitution in (22) leads to the transformed equation

$$\bar{f}(x, y, z, \bar{p}, \bar{q}; t_1, t_2) = 0, \tag{23}$$

the characteristic bands of which for constant values of t_1 and t_2 may be called *trajectory-bands* of $f = 0$.

The case of interest in this connection is associated with the simple transformation of the above form when

$$p = \bar{p} + t_1, \qquad q = \bar{q} + t_2. \tag{24}$$

Referring to the formulas (14) for the conjugate element, we easily see that the conjugate elements of (x, y, z, p, q) and all its transformed elements have also a common point (x', y', z'). For the coördinates x', y' and z' depend upon the partial derivatives of $f = 0$ and these are invariant under (24). We see further from (14) that the conjugate elements are transformed by

$$p' = \bar{p}' - t_1, \qquad q' = \bar{q}' - t_2. \tag{25}$$

Hence if $f(x, y, z, p, q) = 0$ and $f'(x', y', z', p', q') = 0$ are conjugate equations, so also are

$$f(x, y, z, p + t_1, q + t_2) = 0 \quad \text{and} \quad f'(x', y', z', p' - t_1, q' - t_2) = 0.$$

It may be permitted to enlarge upon the geometrical aspect of the facts just presented. Each element (P, E) of $f = 0$ is turned about its point P by the transformation (24), and each of the ∞^2 systems of elements thus obtained is then arranged in the characteristic bands of the corresponding differential equation (23). It is to be noted that the characteristic bands of $f = 0$ do not transform into those of the new equation, for (24) is *not* a contact transformation. If, now, we consider the osculating parabolic bands at (P, E) and at each of its ∞^2 transformed elements, these bands will have in common a *second* point which is common to all the conjugate elements.

A discussion precisely similar applies when the other element coördinates u, v, w, α, β of (19) are employed. Namely, each element is now displaced in its own plane by the transformation

$$a = \bar{a} + \sigma_1, \qquad \beta = \bar{\beta} + \sigma_2, \tag{26}$$

the other coördinates u, v, w remaining constant. Then the conjugate elements of $(u, v, w, \bar{a}, \bar{\beta})$ in the transformed equation, for every σ_1 and σ_2 will also lie in a common plane, namely, the plane of the element $(u', v', w', \alpha', \beta')$ which is conjugate to the initial element (u, v, w, α, β). Furthermore, the conjugate elements satisfy the equations $\alpha' = \bar{\alpha}' - \sigma_1$, $\beta' = \bar{\beta}' - \sigma_2$.

The transformation (26) in the usual coördinates (x, y, z, p, q) takes the form

$$x = \bar{x} - \sigma_1, \qquad y = \bar{y} - \sigma_2, \qquad z = \bar{z} - p\sigma_1 - q\sigma_2, \tag{27}$$

the coördinates p and q remaining constant. Evidently each element is displaced in its own plane in such a manner that the displacement parallel to the XY-plane is the *same* for all elements.

Recapitulation of the results found may be made in the following form:

THEOREM. *Let each surface-element* $(P, E)(x, y, z, p, q)$ *satisfying the differential equation*

$$f(x, y, z, p, q) = 0, \tag{10}$$

be turned about its point in such a manner that it becomes the element (P, E') $(x, y, z, p - t_1, q - t_2)$, *where* t_1 *and* t_2 *are arbitrary constants. Then the transformed elements satisfy the equation*

$$f(x, y, z, p + t_1, q + t_2) = 0,$$

and the osculating parabolic bands of (P, E') *for every* t_1 *and* t_2 *pass through a second common point,*

$$x' = x + \frac{2f_p}{f_s}, \qquad y' = y + \frac{2f_y}{f_z}, \qquad z' = z - 2\,\frac{f_p f_x + f_y f_z}{f_{z^2}},$$

namely, the point of the conjugate element of (P, E) *in* (10).

Again, let each element (P, E) *be moved in its plane so that the transformed element is* $(P', E)(x + \sigma_1, y + \sigma_2, z + p\sigma_1 + q\sigma_2, p, q)$, *where* σ_1 *and* σ_2 *are arbitrary constants. The new elements satisfy the equation*

$$f(x - \sigma_1, y - \sigma_2, z - p\sigma_1 - q\sigma_2, p, q) = 0.$$

The osculating parabolic bands of (P', E) *for every* σ_1 *and* σ_2 *touch a second common plane, namely the plane of the conjugate element of* (P, E) *in* (10).

We may if we so wish consider this theorem as establishing on the parabolic band which osculates a characteristic band at (P, E) the conjugate element. Clearly, as far as the theorem is concerned, the relation of any equation to its conjugate is a mutual one, that is, the new parabolic bands introduced are the *same* for both equations.

Attention may be called to the fact that the ∞^2 parabolic bands issuing from the point (x, y, z) which arise in the first part of the theorem, lie on ∞^1 parabolas, namely, the parabolas through (x, y, z) and the point of the conjugate element. Similarly, the ∞^2 bands referred to in the second part lie on ∞^1 parabolic cylinders.

One final question may be answered. If we start with any element (x, y, z, p, q) satisfying (10), the transformation of the theorem leads to ∞^2 elements at the point (x, y, z). Further, each of these determines a parabolic band, that is, with each is associated a united element. Hence from the discussion of § 1, since each band determines at (x, y, z) ∞^1 curvature elements, we see that from the element (x, y, z, p, q) are derived ∞^3 curvature elements $(x, y, z, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})$. Also there are ∞^1 elements with the same (x, y, z) satisfying (10). Hence for all points we derive a configuration of ∞^7 curvature elements. These must, of course, satisfy a single partial differential equation of the second order. Obviously, we obtain this equation by forming the first partial derivatives of (23) with respect to t_1 and t_2, and then eliminating these parameters.

It should be said that the facts disclosed in the above theorem are easily foreseen in simple properties of the null-system in R_5.

The theorem states, for surface-element loci in space, facts entirely analogous to those established by Scheffers for the plane. The osculating parabolic bands belong also to a second system of differential equations obtained from the conjugate equation by the transformation (25).

The theorem given is stated without great difficulty in a form invariant within the group G_{21} of all contact-transformations under which the system of parabolic bands remains invariant. The ∞^3 points or planes of space transform into a system σ of ∞^3 vertical paraboloids each of which has a parabolic band in common with a fixed vertical paraboloid. Each element of space determines a σ-paraboloid to which it belongs. Then a two parameter group

$$x = \phi(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, t_1, t_2), \cdots, \qquad q = Q(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, t_1, t_2),$$

exists such that each element is displaced upon the corresponding σ-paraboloid. This transformation turns any differential equation

$$f(x, y, z, p, q) = 0$$

into

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q};\ t_1, t_2) = 0.$$

The point now is this. If the conjugate element of (x, y, z, p, q) in $f = 0$ is (x', y', z', p', q') and the corresponding paraboloids are σ and σ', then the conjugate element of $\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}$ in $F = 0$ for every t_1, t_2 will lie upon σ'.

§ 7. *Geometry of Osculating Cubic Bands.*

We now turn our attention to the second type of bands described in § 2. The transformation of Lie used in the previous section is here replaced by the following:

$$(1) \qquad x = \frac{X_1}{Y_1}, \qquad y = \frac{X_2}{Y_2}, \qquad z = Z, \qquad p = Y_1^2, \qquad q = Y_2^2,$$

by which we pass from ordinary space to space of five dimensions R_5. Moreover we verify immediately,

$$(2) \qquad dz - pdx - qdy = dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2,$$

and the null-system of § 4 bears the same relation as before to the geometry of cubic bands. We verify easily that the null-line in R_5,

$$(3) \quad X_1 = X_1^0 + \lambda_1 t,\ X_2 = X_2^0 + \lambda_2 t,\ Y_1 = Y_1^0 + \mu_1 t,\ Y_2 = Y_2^0 + \mu_2 t,\ Z = Z^0 + \nu t,$$

wherein

$$\nu + \mu_1 X_1 - \lambda_1 Y_1 + \mu_2 X_2 - \lambda_2 Y_2 = 0,$$

gives in R_3 a cubic band. For, substituting in (1), we obtain for the equations

of the band corresponding to (3),

$$
\begin{aligned}
&x = x^0 + \frac{(\lambda_1 - \mu_1 x^0)\cdot t}{\sqrt{p^0} + \mu_1 t}, \qquad y = y^0 + \frac{(\lambda_2 - \mu_2 y^0) t}{\sqrt{q^0} + \mu_2 t}, \\
&z = z^0 + [(\lambda_1 - \mu_1 x^0)\sqrt{p^0} + (\lambda_2 - \mu_2 y^0)\sqrt{q^0}]\, t, \qquad p = p^0 + \mu_1 t(2\sqrt{p^0} + \mu_1 t), \\
&q = q^0 + \mu_2 t(2\sqrt{q^0} + \mu_2 t),
\end{aligned} \tag{4}
$$

in which the signs of the radicals are unambiguous. Then clearly, the point locus is a cubic which passes through the points at infinity on the axes of x, y and z. Call these points for convenience A, B and C, respectively. The plane locus is the envelope of

$$Z - z = p(X - x) + q(Y - y), \tag{5}$$

in which x, y, z, p, q have the values in (4). The result of this substitution may be written

$$
\begin{aligned}
(\mu_1^2 X + \mu_2^2 Y - \lambda_1\mu_1 - \lambda_2\mu_2)t^2 + 2[\mu_1\sqrt{p^0}(X - x^0) + \mu_2\sqrt{q^0}(Y - y^0)]t \\
- Z + z^0 + p^0(X - x_0) + q^0(Y - y^0) = 0.
\end{aligned} \tag{6}
$$

The envelope of (6) is a quadric cone. The band (4) has therefore the characteristic described in § 4. The vertex of this cone must lie upon the point-locus, the cubic. It is readily verified that the value of the parameter t which determines the vertex is

$$t_v = \sqrt{p^0 q^0}\,\frac{(\lambda_1 - \mu_1 x^0)\mu_1 + (\lambda_2 - \mu_2 y^0)\mu_2}{\mu_1\mu_2\nu}. \tag{7}$$

The vertex lies at the initial point (x^0, y^0, z^0) when

$$(\lambda_1 - \mu_1 x^0)\mu_1 + (\lambda_2 - \mu_2 y^0)\mu_2 = 0. \tag{8}$$

The cubic band is now the one referred to at the close of § 4.

Pass next to the consideration of what will correspond in R_3 to the general null-plane in R_5,

$$X_1 = aY_1 + bY_2 + c, \qquad X_2 = bY_1 + dY_2 + e, \qquad Z = -cY_1 - eY_2 + f. \tag{9}$$

Eliminating Y_1 and Y_2 from (9) and the first three equations of (1) we obtain as the corresponding surface in R_3

$$\begin{vmatrix} a - x & b & c \\ b & d - y & e \\ -c & -e & f - z \end{vmatrix} = 0. \tag{10}$$

This is the equation of a cubic surface with nodes at the fixed points A, B

and C. Furthermore, it is easily seen that a fourth node exists at the point $(a - be/e, d - be/c, f + ce/b)$.

To the special null-plane

$$X_1 = aY_1, \qquad X_2 = bY_2, \qquad Z = c \tag{11}$$

correspond all the elements of the point (a, b, c).

Two cases of null-planes not representable in the form (9) are

$$Y_1 = aY_2 + b, \qquad X_2 = cY_2 - aX_1 + e, \qquad Z = -eY_2 + bX_1 + h, \tag{12}$$

to which will correspond in R_3 a quadric cone through the fixed points A, B and C; and

$$Y_1 = b, \qquad Y_2 = c, \qquad Z = bX_1 + cX_2 + d, \tag{13}$$

which gives in R_3 the plane $z = b^2x + c^2y + d$.

It may be remarked that (12) is the same as (4), § 6, if $a_1b_2 = a_2^2$, a case to which attention was called at the time, while (13) is also a special case of the same equations. That is, the null-planes in R_5 which under Lie's transformation gave parabolic cylinders now go over into quadric cones. Further, the null-planes (13) in R_5 give planes in R_3 under both transformations.

To the general plane not a null-plane in R_5 will correspond in R_3 an assemblage of ∞^2 elements whose points lie upon a cubic surface having A, B and C as nodes, while the planes envelop a cubic with four nodes, of which the fixed points A, B and C constitute three. This difference in character of plane-locus and point-locus is noteworthy and is readily accounted for by considerations in R_5 which may be omitted here. The three-nodal cubic referred to is entirely *general.*

We may develop as in the previous section the analysis for the theory of cubic bands which osculate the characteristic bands of a given partial differential equation of the first order. To avoid the details, we limit ourselves to the statement of results.

The equations of the cubic band which osculates the characteristic band of

$$f(x, y, z, p, q) = 0$$

at (x, y, z, p, q) are

$$x' = x + \frac{2pf_p t}{p - (f_x + pf_z)t}, \qquad y' = y + \frac{2qf_q t}{q - (f_y + qf_z)t},$$

$$z' = z + 2(pf_p + qf_q)t, \qquad p' = \frac{[p - (f_x + pf_z)t]^2}{p}, \tag{13'}$$

$$q' = \frac{[q - (f_y + qf_z)t]^2}{q}.$$

The osculating cubic bands of the equation

$$f(x, y, z, p, q) = 0 \tag{14}$$

belong also to a second equation obtained by elimination of x, y, z, p, q from (14) and

$$x' = x - 2p\frac{f_p}{f_x}, \qquad y' = y - 2q\frac{f_q}{f_y}, \qquad z' = z + \frac{2pf_p + 2qf_q}{f_z},$$
$$p' = \frac{1}{p}\frac{f_x^2}{f_z^2}, \qquad q' = \frac{1}{q}\frac{f_y^2}{f_z^2}. \tag{15}$$

The conjugate element (x', y', z', p', q') is given by (13′) if $t = 1 \div f_z$.

Finally, we may state in a manner analogous to the theorem at the close of the preceding section the

THEOREM. *Let each surface element* (P, E) (x, y, z, p, q) *of the locus of*

$$f(x, y, z, p, q) = 0 \tag{16}$$

be turned about its point in such a manner that it becomes the element (P, E') $(x, y, z, pe^{-2t_1}, qe^{-2t_2})$, *where* t_1 *and* t_2 *are arbitrary constants. Then the transformed elements satisfy the equation*

$$f(x, y, z, pe^{2t_1}, qe^{2t_2}) = 0,$$

and the osculating cubic bands of (P, E') *for every* t_1 *and* t_2 *pass through a second common point given by the first three equations of* (15).

Again, let each element (P, E) *be moved in its own plane in such a manner that the transformed element* (P', E) *is*

$$(x + \sigma_1/\sqrt{p},\ y + \sigma_2/\sqrt{q},\ z + \sqrt{p}\sigma_1 + \sqrt{q}\sigma_1,\ p, q);$$

then the new elements satisfy the equation

$$f(x - \sigma_1/\sqrt{p}, y - \sigma_2/\sqrt{q}, z - \sqrt{p}\sigma_1 - \sqrt{q}\sigma_2,\ p, q) = 0.$$

Furthermore, the osculating cubic bands of (P', E) *for every* σ_1 *and* σ_2 *will touch a second plane, namely the plane of the element* x', y', z', p', q' *given by* (15).

The statement for cubic bands of the results which correspond to other theorems in the preceding section may be omitted here.

We may regard cubic bands, projectively at least, as space analogues of circles in the plane. For the circle, as a line-element locus, is determined uniquely by two fixed points (the circular points) and two given united elements, while, as we have seen, a cubic band is determined without ambiguity if we postulate that it shall pass through three fixed points and contain two given united surface-elements.

§ 8. *Concluding Theorems.*

Comparison of the transformations of §§ 6 and 7 gives in space the contact-transformation

$$(1)\quad x=\frac{2x'}{p'},\quad y=\frac{2y'}{q'},\quad z=z'-\tfrac{1}{2}p'x'-\tfrac{1}{2}q'y',\quad p=\tfrac{1}{4}p'^2,\quad q=\tfrac{1}{4}q'^2,$$

which has the cardinal property of transforming parabolic into cubic bands. Further properties of this transformation are readily derived from the preceding discussion.

Planes transform into planes.

Parabolic cylinders passing through the extremity of the z-axis at infinity transform into quadric cones having three fixed points A, B, C in common. These points lie at infinity on the axes. Also, straight lines transform into quadric cones of this family.

Vertical paraboloids and parabolas go over into four nodal cubics, three of the nodes being A, B, C. In special cases, the singularities may change to two nodes and one binode, all at A, B and C.

Elimination of p, q, p', q' from (1) leads to the *aequatio-directrix*

$$(2)\qquad xyz-z'xy+y'^2x+x'^2y=0,$$

from which it appears that points (x', y', z') are transformed into cubics for which A and B are nodes and C a binode, while points (x, y, z) go over into vertical paraboloids for which the planes $x'=0$ and $y'=0$ are principal planes.

We may draw one conclusion from the details outlined above. From the double system of generators on a paraboloid we infer at once that for a four-nodal cubic there exists a double system of quadric cones each of which will pass through three of the nodes and touch the cubic along a skew cubic. Taking the four nodes by threes in the four possible ways, we obtain eight systems of enveloping quadric cones.

SHEFFIELD SCIENTIFIC SCHOOL,
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FIELDS OF EXTREMALS IN SPACE

BY

GILBERT AMES BLISS AND MAX MASON*

The minimizing space curves for the integral

$$J = \int F(x, y, z, x', y', z')\, dt$$

form a family which contains four arbitrary constants. Of special interest are the two-parameter families which pass through a given point or are cut transversally by a given curve or surface. Such families occur when the integral J is to be minimized with respect to curves which join two fixed points or which have one end-point fixed while the other is free to vary on a fixed curve or surface.† In the study of the sufficient conditions which insure a minimum for the integral J under these conditions it is essential to know that such two-parameter families fill out simply a portion of space about the point through which all of the curves of the family pass, or about the curve or surface by which they are cut transversally. The minimizing curves, the extremals, are said to form a field in the region in which this property holds.

The field proofs for the analogous cases in the plane have already been made.‡ The proof given by BOLZA for the existence in the plane of a field about an arc of an extremal which does not contain a pair of conjugate points, can be extended readily to the analogous case in space. This proof includes the field proof mentioned above for the extremals which are cut transversally by a given surface.

It is to be shown here that the extremals through a fixed point§ or the extremals to which a given curve is transversal also form fields. In these two cases the functional determinant of the equations to be solved vanishes at the point or along the curve in question. For the former case a suitably selected transformation of the variables reduces the equations to a set whose functional determinant does not vanish. The solution is then obtained from the theory of

* Presented to the Society, Sept. 1, 1908.

† These minimizing problems were considered by the writers in a paper entitled "The properties of curves in space which minimize a definite integral," vol. 9 (1908), p. 440.

‡ See BOLZA, *Vorlesungen über Variationsrechnung*, pp. 164, 249, 271.

§ The construction of a field formed by extremals through a fixed point has been previously discussed by two writers, both of whom used methods quite different from the one given here. See BILL, *The construction of a space field of extremals*, Bulletin of the American Mathematical Society, vol. 15 (1909), p. 374; also HADAMARD, *Leçons sur le calcul des variations*, Note A, p. 497.

implicit functions by the aid of a theorem due to BOLZA,* which is proved in § 1.

A different method is used in § 3 to prove the existence of a field about a curve L, but either this or the method of § 2 is applicable to both problems. In order to apply the method of § 2 to prove the existence of a field about L it is necessary first to construct a system of polar coördinates in the neighborhood of L with properties similar to those of the well-known ones used in § 2.

§ 1. *Bolza's theorem concerning implicit functions.*

Consider a set of equations

$$f_i(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \qquad (i = 1, 2, \cdots, n), \tag{1}$$

for which the functions f_i satisfy the following conditions:

A) The functions f_i are continous and have continuous first partial derivatives with respect to the y's for all values $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ in a certain region $\mathfrak{A}$.

B) There exists a finite closed set $\mathfrak{C}$ of solutions $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ of equation (1) entirely within the region $\mathfrak{A}$. The projection of $\mathfrak{C}$ in the $(x_1, x_2, \cdots, x_m)$-space will be denoted by $\mathfrak{H}$.

C) No two distinct points in $\mathfrak{C}$ have the same projection in $\mathfrak{H}$: in other words, to any point $(x_1, x_2, \cdots, x_m)$ in $\mathfrak{H}$ there corresponds one and but one solution $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ in $\mathfrak{C}$.

D) In the set $\mathfrak{C}$ the functional determinant $\partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ is always different from zero.

Then there exists a constant ϵ such that no two solutions of equations (1) in $\mathfrak{C}_\epsilon$† possess the same projection in $\mathfrak{H}_\epsilon$, and a constant $\delta \leqq \epsilon$ can be determined so that to any point in the neighborhood $\mathfrak{H}_\delta$ there corresponds at least one solution of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$. The functions

$$y_1(x_1, x_2, \cdots, x_m),\ y_2(x_1, x_2, \cdots, x_m),\ \cdots,\ y_n(x_1, x_2, \cdots, x_m) \tag{2}$$

so defined in $\mathfrak{H}_\delta$ are of class C, and if the function f_i are of class $C^{(n)}$‡ in $\mathfrak{A}$ these functions are also of class $C^{(n)}$ in $\mathfrak{H}_\delta$.

* Loc. cit., p. 166. BOLZA makes the proof of the general theorem depend upon a special case which he has discussed in the preceding pages. The theorem was proved independently by the writers for the purposes of the present paper, and the proof in § 1 is the direct proof which they derived, the statement of the theorem being slightly different from that of Bolza.

† See BOLZA, loc. cit., p. 154. His notation is $(\varepsilon)_{\mathfrak{C}}$. The neighborhood $\mathfrak{C}_\epsilon$ for any set $\mathfrak{C}$ and any constant ε is the set of points $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ for each of which there exists at least one point $(x_1', x_2', \cdots, x_m' : y_1', y_2', \cdots, y_n')$ in $\mathfrak{C}$ satisfying the relations

$$|x_1 - x_1'| < \varepsilon,\ |x_2 - x_2'| < \varepsilon,\ \cdots,\ |x_m - x_m'| < \varepsilon,$$

with similar inequalities for $y_1, y_2, \cdots, y_n$.

‡ A function is of class $C^{(n)}$ if its derivatives up to and including those of order n are continuous.

As a first step in the proof it can be shown that a constant ϵ exists such that no two distinct solutions of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$ can have the same projection $(x_1, x_2, \cdots, x_m)$ in $\mathfrak{H}_\epsilon$. For if this were not the case two sequences of solutions of equations (1)*

$$(3)\qquad \begin{gathered} \{x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)} : y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}\}, \\ \{x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)} : Y_1^{(i)}, Y_2^{(i)}, \cdots, Y_n^{(i)}\} \qquad (i=1, 2, \cdots), \\ (y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}) \neq (Y_1^{(i)}, Y_2^{(i)}, \cdots, Y_n^{(i)}), \end{gathered}$$

could be found, with limit points

$$(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n), \qquad (\xi_1, \xi_2, \cdots, \xi_m : H_1, H_2, \cdots, H_n)$$

which lie in the region $\mathfrak{C}$. Since by C) no two points in $\mathfrak{C}$ have the same projection in $\mathfrak{H}$ it follows that $(\eta_1, \eta_2, \cdots, \eta_n)$ and $(H_1, H_2, \cdots, H_n)$ are the same. At the point $(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n)$ the usual theorems on implicit functions† can be applied to equations (1) on account of A), B), D), and it follows that to any point in the neighborhood of $(\xi_1, \xi_2, \cdots, \xi_m)$ there can correspond but one set of solutions in the neighborhood of $(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n)$. Thus the existence of the sequences (3) is contradicted.

The constant ϵ can be still further restricted if necessary so that the functional determinant $\partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ is different from zero in $\mathfrak{C}_\epsilon$, on account of D).

The next step is to select a constant $\delta \leqq \epsilon$ so that to each point in $\mathfrak{H}_\delta$ there corresponds at least one solution in the region $\mathfrak{C}_\epsilon$. Suppose that P is any point of the set $\mathfrak{C}$, and Q its projection in $\mathfrak{H}$. Then by the theorems on implicit functions referred to above, a constant δ can be found such that to each point $(x_1, x_2, \cdots, x_m)$ in the neighborhood Q_δ there corresponds a solution of equations (1) lying in P_ϵ and therefore in $\mathfrak{C}_\epsilon$. A constant δ can be selected which will be effective in this way uniformly for all the points P of $\mathfrak{C}$. Otherwise a sequence $\{P_n\}$ would exist in $\mathfrak{C}$, with limit point π, for which the corresponding constants δ_n would have the limit zero. This is, however, impossible, since there is a value $\delta_\pi > 0$ for the limit point π.

By combining these results it follows that to any point in the region $\mathfrak{H}_\delta$ there corresponds one and but one solution of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$. Since the functional determinant does not vanish in $\mathfrak{C}_\epsilon$ it follows from the usual theorems concerning implicit functions that the functions (2) so defined over the region $\mathfrak{H}_\delta$ will be surely of class C under the conditions on the functions f_i stated under A), and of class $C^{(n)}$ when the functions f_i are of class $C^{(n)}$ in $\mathfrak{A}$.

* For the details of the construction of these sequences see BOLZA, loc. cit., p. 161.

† See, for example, OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, p. 52; or GOURSAT-HEDRICK, *A Course in Analysis*, vol. 1, p. 45 and footnote.

§ 2. *The field about a point.*

The equations of the extremals through a given point $P(x_0, y_0, z_0)$ may be taken in the form *

$$x = \phi(s, \xi, \eta, \zeta), \qquad y = \psi(s, \xi, \eta, \zeta), \qquad z = \chi(s, \xi, \eta, \zeta),$$

where the functions ϕ, ψ, χ and their first partial derivatives with respect to s are of class C' in the neighborhood of the values of s, ξ, η, ζ defined by the conditions

$$(\mathfrak{S}) \qquad s = 0, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

For $s = 0$ the identities

$$(4) \quad \begin{aligned} &\phi(0, \xi, \eta, \zeta) = x_0, \qquad \psi(0, \xi, \eta, \zeta) = y_0, \qquad \chi(0, \xi, \eta, \zeta) = z_0, \\ &\phi_s(0, \xi, \eta, \zeta) = \xi, \qquad \psi_s(0, \xi, \eta, \zeta) = \eta, \qquad \chi_s(0, \xi, \eta, \zeta) = \zeta \end{aligned}$$

hold, and the identity

$$\phi\phi_s + \psi\psi_s + \chi\chi_s = 0$$

is satisfied for all arguments. If

$$(5) \qquad \xi^2 + \eta^2 + \zeta^2 = 1,$$

it follows that s is the length of arc measured along the extremal from the point P, so that

$$(6) \qquad \phi_s^2 + \psi_s^2 + \chi_s^2 = 1,$$

and ξ, η, ζ are the direction cosines of the tangent to the extremal at P.

It is desired to show that *two positive constants d, e exist such that to any point (x, y, z) in the region P_d'*,†

$$(P_d') \qquad 0 < |x - x_0| < d, \ 0 < |y - y_0| < d, \ 0 < |z - z_0| < d,$$

there exists one and but one solution $(x, y, z : s, \xi, \eta, \zeta)$ of the equations

$$(7) \quad \xi^2 + \eta^2 + \zeta^2 = 1, \ \phi(s, \xi, \eta, \zeta) = x, \ \psi(s, \xi, \eta, \zeta) = y, \ \chi(s, \xi, \eta, \zeta) = z$$

for which (s, ξ, η, ζ) is in the region $\mathfrak{S}_e'$ defined by the conditions

$$(\mathfrak{S}_e') \qquad 0 < s < e, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

In terms of the polar coördinates R, A, B, C defined by the equations

$$x - x_0 = RA, \ y - y_0 = RB, \ z - z_0 = RC, \ R > 0, \ A^2 + B^2 + C^2 = 1,$$

* See MASON and BLISS, loc. cit., p. 444.

† Note that P_d' is the region P_d as defined in a preceding footnote, with the point (x_0, y_0, z_0) excluded.

the equations (7) become

$$(8)\quad \xi^2+\eta^2+\zeta^2=1,\ \phi(s,\xi,\eta,\zeta)=RA,\ \psi(s,\xi,\eta,\zeta)=RB,\ \chi(s,\xi,\eta,\zeta)=RC.$$

If s is replaced by σ, where $s=R\sigma$, they take the form

$$(9)\quad \begin{aligned} &\xi^2+\eta^2+\zeta^2=1,\\ &\frac{1}{R}[\phi(R\sigma,\xi,\eta,\zeta)-x_0]=A,\\ &\frac{1}{R}[\psi(R\sigma,\xi,\eta,\zeta)-y_0]=B,\\ &\frac{1}{R}[\chi(R\sigma,\xi,\eta,\zeta)-z_0]=C. \end{aligned}$$

By Taylor's formula with the integral form of the remainder term * and the identities (4), it follows that

$$(10)\quad \phi(s,\xi,\eta,\zeta)-x_0=s\int_0^1\phi_s(ts,\xi,\eta,\zeta)\,dt,$$

with similar equations for ψ and χ, so that when the first members of equations (9) are defined for $R=0$ by their limiting values, these equations reduce, for $R=0$, to

$$(11)\quad \xi^2+\eta^2+\zeta^2=1,\ \sigma\xi=A,\ \sigma\eta=B,\ \sigma\zeta=C.$$

It is evident then that equations (9) have the solutions

$$(\mathfrak{C})\quad (R,A,B,C:\sigma,\xi,\eta,\zeta)=(0,A,B,C:1,\xi,\eta,\zeta),\ A^2+B^2+C^2=1,$$

and the projection of $\mathfrak{C}$ in terms of the polar coördinates is the region defined by the conditions

$$(\mathfrak{H})\quad R=0,\qquad A^2+B^2+C^2=1.$$

From equation (10) and its analogues for ψ and χ, it follows that the left members of equations (9) are of class C' in the neighborhood of $\mathfrak{C}$, and in $\mathfrak{C}$ their functional determinant with respect to σ,ξ,η,ζ is calculated from (11) to be $2\sigma^2=2$. Hence Bolza's theorem is applicable, and it follows that a constant ϵ exists such that no two solutions of equations (9) in the region $\mathfrak{C}_\epsilon$ have the same projection in $\mathfrak{H}_\epsilon$, and a second constant $\delta\leqq\epsilon$, can be chosen so that to any (R,A,B,C) in $\mathfrak{H}_\delta$ there corresponds a set of values

$$(12)\quad \sigma(R,A,B,C),\ \xi(R,A,B,C),\ \eta(R,A,B,C),\ \zeta(R,A,B,C)$$

which with (R,A,B,C) satisfy equations (9) and lie in $\mathfrak{C}_\epsilon$. The functions (12) are of class C' and reduce to $1,A,B,C$ for $R=0$.

* See C. JORDAN, *Cours d'Analyse*, vol. 1, p. 245.

With these results in mind it is possible to show the existence of a constant e such that no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ can define by means of equations (7) the same point (x, y, z), or what is equivalent, the same values (R, A, B, C). In fact, for any solution of equations (8) the value of R is given in terms of (s, ξ, η, ζ) by means of the formula

$$R^2 = [\phi - x_0]^2 + [\psi - y_0]^2 + [\chi - z_0]^2,$$

and from equations (10) and (4) it is evident therefore that the quotient s/R is a continuous function of s, ξ, η, ζ which reduces to 1 for $s = 0$. Similarly A, B, C in equations (8) are seen to be continuous functions of s, ξ, η, ζ which take the values ξ, η, ζ, respectively, for $s = 0$. Hence if e is sufficiently small, no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ can define the same (x, y, z). Otherwise equations (9) would have two solutions $(R, A, B, C : \sigma, \xi, \eta, \zeta)$ in $\mathfrak{C}$ corresponding to the same (R, A, B, C).

Let δ be taken so small that the values

$$(13)\quad s = R\sigma(R, A, B, C),\ \xi(R, A, B, C),\ \eta(R, A, B, C),\ \zeta(R, A, B, C)$$

corresponding to any (R, A, B, C) in $\mathfrak{H}_\delta$, lie in $\mathfrak{S}_e$. If $d = \delta/\sqrt{3}$, then to any point (x, y, z) in P'_d there will correspond values of (R, A, B, C) in $\mathfrak{H}_\delta$, and consequently a solution of equations (7) for which (s, ξ, η, ζ) lies in $\mathfrak{S}'_e$. By the preceding paragraph this solution will be unique in $\mathfrak{S}'_e$.

The functions

$$(14)\qquad s(x, y, z),\ \xi(x, y, z),\ \eta(x, y, z),\ \zeta(x, y, z)$$

so defined have the following properties:

1) *In the neighborhood P'_d all are of class C'.*

2) *In P_d the function $s(x, y, z)$ is continuous and has the value 0 at (x_0, y_0, z_0).*

3) *Let C be an arc of class C',*

$$(C)\qquad x = f(t),\quad y = g(t),\quad z = h(t),\qquad (0 \leqq t \leqq T),$$

which passes through $P(x_0, y_0, z_0)$ for $t = 0$. The function $s(t)$ defined by substituting the values (C) in the expressions (14) has at $t = 0$ the derivative

$$s'(0) = \sqrt{f'^2(0) + g'^2(0) + h'^2(0)},$$

and the corresponding functions $\xi(t), \eta(t), \zeta(t)$ are continuous and reduce for $t = 0$ to the direction cosines A_0, B_0, C_0, of the positive tangent to C at P, respectively.

The first two properties follow immediately from the nature of the functions

(2) and of the polar coördinates R, A, B, C as functions of x, y, z. For points on the curve C

$$R^2 = [f - x_0]^2 + [g - y_0]^2 + [h - z_0]^2,$$

$$A = \frac{f - x_0}{R}, \qquad B = \frac{g - y_0}{R}, \qquad C = \frac{h - z_0}{R}.$$

Hence by applying Taylor's formula to the differences $f - x_0$, $g - y_0$, $h - z_0$ it follows that

$$\lim_{t=0} \frac{R}{t} = \sqrt{f'^2(0) + g'^2(0) + h'^2(0)}$$

and also that A, B, C approach the limits A_0, B_0, C_0. From the equations (13) and the properties of the functions (12) it is then easy to prove the statement 3).

The proof which has been made applies to the construction of a field in the neighborhood of a single point (x_0, y_0, z_0). It is important, for the proof of Hilbert's theorem concerning the existence of an extremal joining two given points in space, to know that such fields can be constructed uniformly about the points of a finite closed region $\mathfrak{R}$. The equations of the extremals through points of $\mathfrak{R}$ have the form *

$$\begin{aligned} x &= \phi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \\ y &= \psi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \\ z &= \chi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \end{aligned} \tag{15}$$

where ϕ, ψ, χ and their derivatives for s are of class C' in the neighborhood of the domain defined by the conditions

$$(x_0, y_0, z_0) \text{ in } R, \ s = 0, \ \xi^2 + \eta^2 + \zeta^2 = 1.$$

For $s = 0$ the identities (4) hold with respect to all seven arguments, and the identity (6) is a consequence of equation (5).

The equations corresponding to (9) can be set up in the same manner as before, but they contain now the additional arguments x_0, y_0, z_0. The region corresponding to $\mathfrak{E}$ for these equations is the region

$$(\bar{\mathfrak{E}}) \quad (x_0, y_0, z_0 : 0, A, B, C : 1, A, B, C), \ (x_0, y_0, z_0) \text{ in } R, \ A^2 + B^2 + C^2 = 1,$$

and the region corresponding to $\mathfrak{H}$ is

$$(\bar{\mathfrak{H}}) \quad (x_0, y_0, z_0 : 0, A, B, C), \ (x_0, y_0, z_0) \text{ in } R, \ A^2 + B^2 + C^2 = 1.$$

* MASON AND BLISS, loc. cit., p. 444.

The constants δ and ϵ can be determined so that no two solutions of equations (9) in $\bar{\mathfrak{C}}_\epsilon$, have the same projection in $\bar{\mathfrak{H}}_\epsilon$, and so that to each point in $\bar{\mathfrak{H}}_\delta$ there corresponds at least one solution in $\bar{\mathfrak{C}}_\epsilon$, the variables σ, ξ, η, ζ being now functions of class C' of $x_0, y_0, z_0, R, A, B, C$ in $\bar{\mathfrak{H}}_\delta$ which reduce to $(1, A, B, C)$ for points in $\bar{\mathfrak{H}}$.

Each solution $(x_0, y_0, z_0 : R, A, B, C : s, \xi, \eta, \zeta)$ of equations (8) defines as before a solution $(x_0, y_0, z_0 : R, A, B, C : \sigma, \xi, \eta, \zeta)$ of equations (9) whose elements are continuous functions of $x_0, y_0, z_0, s, \xi, \eta, \zeta$ reducing to $(x_0, y_0, z_0 : 0, A, B, C : 1, \xi, \eta, \zeta)$ when (x_0, y_0, z_0) is in R and (s, ξ, η, ζ) is in the region $\mathfrak{S}$. Hence the constant e can be chosen so that the solution of equation of equation (9) always lies in $\bar{\mathfrak{C}}_\epsilon$ when (x_0, y_0, z_0) is in $\mathfrak{R}$ and (s, ξ, η, ζ) in $\mathfrak{S}'_e$, and consequently for any fixed but arbitrarily chosen point (x_0, y_0, z_0) in $\mathfrak{R}$ no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ could define the same (R, A, B, C). Otherwise equations (9) would have two solutions in $\bar{\mathfrak{C}}_\epsilon$ corresponding to the same point in $\bar{\mathfrak{H}}_\epsilon$.

The constant $d = \delta/\sqrt{3}$ can similarly be chosen so that for any point P (x_0, y_0, z_0) in $\mathfrak{R}$ the values (R, A, B, C) in the region P'_d about P define solutions $(x, y, z : s, \xi, \eta, \zeta)$ of equations (15) and (5) for which (s, ξ, η, ζ) is unique in $\mathfrak{S}'_e$.

Hence constants d, e can be chosen uniformly for the region $\mathfrak{R}$ so that to any point (x, y, z) in the neighborhood P'_d,

$$0 < |x - x_0| < d, \qquad 0 < |y - y_0| < d, \qquad 0 < |z - z_0| < d,$$

of a point P (x_0, y_0, z_0) of $\mathfrak{R}$ there exists one and but one solution of equations (15) for which (s, ξ, η, ζ) is in the region

$$0 < s < e, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

The functions of x_0, y_0, z_0, x, y, z so determined are of class C' for (x_0, y_0, z_0) in $\mathfrak{R}$ and (x, y, z) in P'_d, and have the properties described above when (x_0, y_0, z_0) is fixed.

From this result is derived the following geometrical theorem:

A constant $\delta > 0$ can be determined so that through any point $Q \neq P$ in a sphere of radius δ about P, where P is an arbitrarily selected point of R, there passes one and but one extremal arc (15) which joins P with Q and lies entirely within the sphere.

Along an extremal, R is a continuous function of $x_0, y_0, z_0, s, \xi, \eta, \zeta$ and has a derivative dR/ds which reduces to 1 for $s = 0$. To make sure that the extremal arc joining P with Q lies entirely within the sphere it is therefore only necessary to choose e so small that dR/ds remains different from zero for (x_0, y_0, z_0) in R and (s, ξ, η, ζ) in the region S'_e. Then the corresponding constant δ will be the one desired.

§ 3. *The construction of a field of extremals to which a given arc* $\mathfrak{L}$ *is transversal.*

Consider an arc $\mathfrak{L}$

$$(\mathfrak{L}) \qquad x = X(u),\ y = Y(u),\ z = Z(u),\ u_1 \leqq u \leqq u_2,$$

which has the properties that it is part of a curve of class C'', that it does not intersect itself, and that the parameter u is the length of arc. Let

$$(1) \qquad \alpha, \beta, \gamma : l, m, n : \lambda, \mu, \nu$$

be the direction cosines of the tangent, principal normal, and binormal, respectively.

In each half-plane through a tangent to $\mathfrak{L}$ and containing the direction

$$l \cos v + \lambda \sin v,\ m \cos v + \mu \sin v,\ n \cos v + \nu \sin v$$

normal to the curve, there is one and but one direction ξ, η, ζ distinct from the

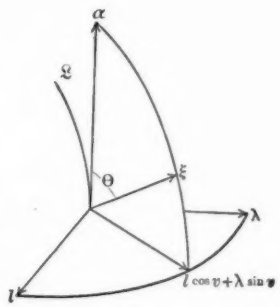

FIG. 1.

tangent, to which the curve $\mathfrak{L}$ is transversal.* The angle $\theta (0 < \theta < \pi)$ (see Fig. 1) determining this direction, is a function of u, v which is periodic in v with period 2π, and the values of ξ, η, ζ are

$$(2) \qquad \begin{aligned} \xi &= \alpha \cos\theta + (l \cos v + \lambda \sin v) \sin\theta, \\ \eta &= \beta \cos\theta + (m \cos v + \mu \sin v) \sin\theta, \\ \zeta &= \gamma \cos\theta + (n \cos v + \nu \sin v) \sin\theta. \end{aligned}$$

The extremals to which $\mathfrak{L}$ is transversal are found by substituting the functions $X(u), Y(u), Z(u)$ defining $\mathfrak{L}$, in place of x_0, y_0, z_0 in equations (15) of

*BLISS AND MASON, loc. cit., p. 463.

§ 2, and the values (2) in place of ξ, η, ζ. Then the equations of the extremals have the form

$$(E) \qquad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

where the functions ϕ, ψ, χ can be supposed of class C'' in a certain neighborhood of the region $\mathfrak{S}$ defined by the conditions

$$(\mathfrak{S}) \qquad s = 0, \qquad u_1 \leqq u \leqq u_2, \qquad v \text{ any value},$$

and are periodic in v with period 2π. The parameter s is the length of arc, so that the identity

$$\phi_s^2 + \psi_s^2 + \chi_s^2 \equiv 1$$

is satisfied for all arguments. For $s = 0$,

$$(3)\quad \phi(0, u, v) \equiv X(u), \qquad \psi(0, u, v) \equiv Y(u), \qquad \chi(0, u, v) \equiv Z(u),$$

$$(4)\quad \begin{aligned} \phi_s(0, u, v) &\equiv \alpha \cos\theta + (l \cos v + \lambda \sin v) \sin\theta, \\ \psi_s(0, u, v) &\equiv \beta \cos\theta + (m \cos v + \mu \sin v) \sin\theta, \\ \chi_s(0, u, v) &\equiv \gamma \cos\theta + (n \cos v + \nu \sin v) \sin\theta, \end{aligned}$$

and since θ is between 0 and π, it follows from the relations which are satisfied by the mutually orthogonal directions (1) that

$$\cos\theta = \alpha\phi_s(0, u, v) + \beta\psi_s(0, u, v) + \gamma\chi_s(0, u, v),$$

$$\begin{aligned} \sin\theta = (l \cos u + \lambda \sin v)\phi_s(0, u, v) &+ (m \cos v + \mu \sin v)\psi_s(0, u, v) \\ &+ (n \cos v + \nu \sin v)\chi_s(0, u, v) > 0. \end{aligned}$$

If A, B, C are the cosines of a direction not coincident with the tangent to $\mathfrak{L}$, then there is one and but one value of v (modulo 2π) for which the tangent to the extremal E is coplanar with A, B, C and the tangent to $\mathfrak{L}$, and which lies on the same side of the tangent to $\mathfrak{L}$ with A, B, C. For in order that these conditions may be satisfied the determinant

$$\begin{vmatrix} \phi_s(0, u, v) & \alpha & A \\ \psi_s(0, u, v) & \beta & B \\ \chi_s(0, u, v) & \gamma & C \end{vmatrix}$$

must vanish, and the expression

$$\cos\omega = A(l \cos v + \lambda \sin v) + B(m \cos v + \mu \sin v) + C(n \cos v + \nu \sin v)$$

must be positive. With the help of equations (4) the former condition is seen to determine the ratio of $\cos v$ and $\sin v$, and the latter determines their signs,

uniquely. It follows that if v and v' are not congruent modulo π the determinant

$$(5)\qquad \begin{vmatrix} \phi_s(0,u,v) & \alpha & \phi_s(0,u,v') \\ \psi_s(0,u,v) & \beta & \psi_s(0,u,v') \\ \chi_s(0,u,v) & \gamma & \chi_s(0,u,v') \end{vmatrix} = \sin(v-v')$$

is always different from zero.

By substitution of the values for ϕ_s and ϕ_{sv} from equations (4) a simple computation shows that when $v \equiv v'$ (modulo π)

$$(6)\qquad \begin{vmatrix} \phi_s(0,u,v) & \alpha & \phi_{sv}(0,u,v') \\ \psi_s(0,u,v) & \beta & \psi_{sv}(0,u,v') \\ \chi_s(0,u,v) & \gamma & \chi_{sv}(0,u,v') \end{vmatrix} = \pm \sin\theta \sin\theta' \neq 0.$$

The notation $\mathfrak{L}_\delta$ will be used to denote the δ-neighborhood of the curve $\mathfrak{L}$, and $\mathfrak{L}'_\delta$ will denote the points of this set exclusive of those on the curve of which $\mathfrak{L}$ is a part. In the following pages any two points (s, u, v) and (s', u', v') are regarded as identical when $s = s'$, $u = u'$, and v is congruent to v' modulo 2π $(v \equiv v')$, otherwise distinct. The region $\mathfrak{S}_\epsilon$ is

$$(\mathfrak{S}_\epsilon)\qquad 0 \leqq s \leqq \epsilon, \qquad u_1 - \epsilon \leqq u \leqq u_2 + \epsilon, \qquad v \text{ any value},$$

and $\mathfrak{S}'_\epsilon$ is $\mathfrak{S}_\epsilon$ exclusive of the points for which $s = 0$.

The purpose of the present section is the proof of the following statement:

Two positive constants δ, ϵ can be determined so that to any point (x, y, z) in the neighborhood $\mathfrak{L}'_\delta$ there corresponds one and but one solution $(x, y, z : s, u, v)$ of the equations (E) for which the values (s, u, v) lie in the region $\mathfrak{S}'_\epsilon$.

In the first place ϵ can be chosen so small that no two points (s, u, v), (s', u', v') in the region $\mathfrak{S}_\epsilon$ define the same point (x, y, z) by means of the equations (E), unless $s = s' = 0$, $u = u'$. In other words, no two of the extremals (E) can intersect for values of (s, u, v) in $\mathfrak{S}_\epsilon$ unless they pass through the same point P of the curve of which $\mathfrak{L}$ is a part, in which case P is the only point of intersection, and furthermore no extremal intersects itself. For if these statements were not true, a sequence

$$(7)\qquad \{s_i, u_i, v_i : s'_i, u'_i, v'_i\} \qquad (i = 1, 2, \cdots),$$
$$(s_i, s'_i) \neq (0, 0), \qquad (s_i, u_i, v_i) \neq (s'_i, u'_i, v'_i)$$

of solutions of the equations

$$(8)\qquad \begin{aligned} \phi(s,u,v) &= \phi(s',u',v'), \\ \psi(s,u,v) &= \phi(s',u',v'), \\ \chi(s,u,v) &= \chi(s',u',v'), \end{aligned}$$

could be determined with limiting values of the form

$$(9) \qquad (0, u_0, v_0 : 0, u_0', v_0'), \qquad u_1 \leqq u_0 \leqq u_2, \qquad u_1 \leqq u_0' \leqq u_2,$$

which would also satisfy the equations. Since $\mathfrak{L}$ does not intersect itself, it follows from the identities (3) that u_0 and u_0' must be the same.

With the help of the identities (3) any one of the equations (8), say the first, can be written in the form

$$[\phi(s, u, v) - \phi(0, u', v)] - [\phi(s', u', v') - \phi(0, u', v')] = 0,$$

or by Taylor's formula

$$s\int_0^1 \phi_s[ts, u' + t(u-u'), v]dt + (u-u')\int_0^1 \phi_u[ts, u' + t(u-u'), v]dt - s'\int_0^1 \phi_s(ts', u', v')dt = 0.$$

The three equations so found would have solutions s_i, $u_i - u_i'$, s_i' not all zero provided by the sequence (7). Their determinant with respect to the arguments s, $u-u'$, s' outside the integral signs, is a continuous function of s, u, v, s', u', v' which must vanish for the arguments (7), and therefore also for the arguments (9). But for the latter it takes the form (5), showing that v_0 must be congruent to v_0' modulo π, since the determinant (5) vanishes for such values only.

From the identities (3) it appears that the function $\phi_s(0, u, v)$ is identically zero, and therefore

$$(10) \qquad \phi_s(s, u, v) = s\int_0^1 \phi_{ss}(\tau s, u, v)d\tau.$$

By an application of Taylor's formula, and then of equation (10), the first of equations (8) can be written again in the form

$$\begin{aligned}\phi(s,u,v)-\phi(s',u',v') &= \phi(s,u,v) - \phi(s',u',v) + \phi(s',u',v) - \phi(s',u',v') \\ &= (s-s')\int_0^1 \phi_s[s' + t(s-s'), u' + t(u-u'), v]dt \\ &\quad + (u-u')\int_0^1 \phi_u[s' + t(s-s'), u' + t(u-u'), v)]dt \\ &\quad + (v-v')s'\int_0^1\int_0^1 \phi_{ss}[\tau s', u', v' + t(v-v')]d\tau dt.\end{aligned}$$

The three equations so found have a determinant with respect to the quantities $s-s'$, $u-u'$, $(v-v')s'$ outside the integral signs, which is a continuous function reducing to (6) for the arguments $(0, u, v : 0, u, v')$. At the limiting values (9) this determinant is different from zero since $u_0 = u_0'$, $v_0 \equiv v_0'$ (modulo π).

Hence for a sufficiently large value of i any quantities $s_i - s_i'$, $u_i - u_i'$, $(v_i - v_i')s_i'$ which satisfy these equations must all vanish, and by interchanging (s, u, v) with (s', u', v') in equation (8) it follows that $(v_i' - v_i)s_i$ must also be zero. If the sequence (7) existed, therefore, the points (s_i, u_i, v_i) and (s_i', u_i', v_i') would have to coincide, or else $s_i = s_i'$, $u_i = u_i'$, and s_i and s_i' would have to vanish simultaneously, for each sufficiently large value of i. But these are the properties which the sequence (7) was assumed not to have, and consequently its existence is impossible.

The constant ϵ can be subjected to a second restriction, if necessary, to the effect that for all values of (s, u, v) in $\mathfrak{S}_\epsilon'$ the functional determinant of equations (E) with respect to s, u, v is different from zero. With the help of equation (10) this determinant takes the form

$$\begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix} = sV(s, u, v),$$

where V is a continuous function in the region $\mathfrak{S}_\epsilon$, different from zero for $s = 0$ on account of the condition (6). Hence if ϵ is properly chosen V will remain different from zero in $\mathfrak{S}_\epsilon'$.

Suppose now that ϵ has been chosen so as to satisfy the two restrictions described above. The points (x, y, z) of the extremals (E) for which $u = u_1 - \epsilon$ or $u = u_2 + \epsilon$, or for which $s = \epsilon$, form a closed set which has no point in common with the arc $\mathfrak{L}$. Let $\delta\sqrt{3}$ be the minimum distance between the points of this set and the points of $\mathfrak{L}$. Then any solution $(x, y, z : s, u, v)$ of equations (E), corresponding to a point $P(x, y, z)$ in $\mathfrak{L}_\delta'$, has values (s, u, v) in the *interior* of $\mathfrak{S}_\epsilon'$. For P is not on $\mathfrak{L}$ and therefore can not correspond to a set (s, u, v) for which $s = 0$, and its distance from $\mathfrak{L}$ is less than $\delta\sqrt{3}$, so that, on account of the way in which δ has just been selected, no one of the three conditions $u = u_1 - \epsilon$, $u = u_3 + \epsilon$, $s = \epsilon$ can be satisfied.

From the first property of the constant ϵ it follows, furthermore, that P can not correspond to more than one set (s, u, v) in $\mathfrak{S}_\epsilon'$.

On the other hand, each point P in $\mathfrak{L}_\delta'$ corresponds to at least one set of values (s, u, v) in the interior of $\mathfrak{S}_\epsilon$. For consider any particular point $P'(x', y', z')$ in $\mathfrak{L}_\delta'$ which has corresponding to it the values (s', u', v'). Join P' with P by an arc D

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad t' \leqq t \leqq T$$

of class C' which lies entirely within $\mathfrak{L}_\delta'$. The functional determinant with respect to s, u, v of the first members of the equations

$$(11)\quad \phi(s, u, v) = x(t), \qquad \psi(s, u, v) = y(t), \qquad \chi(s, u, v) = z(t)$$

is different from zero at (s', u', v') on account of the second property of the constant ϵ, and these equations have the particular solutions (x', y', z', t'). It follows that equations (11) have continuous solutions $s(t), u(t), v(t)$ in the neighborhood of t', which can be continued along the arc D up to the point P, so that P also corresponds to a solution $(x, y, z : s, u, v)$ of equations (E) for which (s, u, v) necessarily lies in $\mathfrak{S}'_\epsilon$. Otherwise there would be an upper bound τ to the values of t which could be reached by continuation, with a corresponding solution $(s_\tau, u_\tau, v_\tau, \tau)$ of equations (11) for which (s_τ, u_τ, v_τ) would lie in $\mathfrak{S}'_\epsilon$. The functional determinant of equations (11) would still be different from zero at (s_τ, u_τ, v_τ) and consequently τ could not be the upper bound described.

It has therefore been shown that to each point (x, y, z) in $\mathfrak{L}'_\delta$ there corresponds a solution $(x, y, z : s, u, v)$ of equations (E) for which (s, u, v) lies in $\mathfrak{S}'_\epsilon$, and on account of the way in which ϵ was chosen this solution is unique.

The functions

$$s(x, y, z),\ u(x, y, z),\ v(x, y, z) \tag{12}$$

which with x, y, z solve equations (E) have the following properties in the region $\mathfrak{L}'_\delta$:

1) *They are all three of class C' in $\mathfrak{L}'_\delta$.*

2) *In $\mathfrak{L}_\delta$ the first two are continuous and for a point (x_0, y_0, z_0) on $\mathfrak{L}$ corresponding to the parameter value u_0,*

$$s(x_0, y_0, z_0) = 0, \qquad u(x_0, y_0, z_0) = u_0.$$

3) *The function $v(x, y, z)$ is infinitely many valued, its values corresponding to a given (x, y, z) being all congruent modulo 2π, but $\cos v$ and $\sin v$ are single valued functions*

$$\cos v = a(x, y, z), \qquad \sin v = b(x, y, z) \tag{13}$$

of class C' in L'_δ.

4) *Let C be a curve*

$$(C) \qquad x = f(t), \qquad y = g(t), \qquad z = h(t) \qquad (0 \leqq t \leqq T),$$

with a continuously turning tangent, which intersects $\mathfrak{L}$ at a point P (x_0, y_0, z_0, u_0) for $t = 0$, but is not tangent to $\mathfrak{L}$ at P, and for which t is the length of arc. The tangent to C at P determines uniquely a direction

$$l \cos v_0 + \lambda \sin v_0, \qquad m \cos v_0 + \mu \sin v_0, \qquad n \cos v_0 + \nu \sin v_0, \tag{14}$$

coplanar with the tangents to C and $\mathfrak{L}$ at P, and lying on the same side of the tangent to $\mathfrak{L}$ with the positive tangent to C. As t approaches zero the functions $s(t), u(t), a(t), b(t)$ defined by substituting the values of x, y, z

from (C) *in the functions* (12) *and* (13), *approach the limits* 0, u_0, $\cos v_0$, $\sin v_0$, *and the functions* $s(t)$, $u(t)$ *have derivatives at* $t=0$ *which have the values*

$$s'(0)=\sin\omega\csc\theta, \qquad u'(\theta)=\cos\omega-\sin\omega\cot\theta,$$

where ω *is the angle between the positive tangents to* C *and* $\mathfrak{L}$, *and* θ *is the corresponding angle for* E *and* $\mathfrak{L}$.

The properties 1) and 3) follow readily from the usual theorems concerning implicit functions and the periodicity of ϕ, ψ, χ with respect to v. For at any solution $(x', y', z' : s', u', v')$ of equations (E) for which (x', y', z') is in $\mathfrak{L}'_\delta$ and (s', u', v') is in $\mathfrak{S}'_\epsilon$, the functional determinant of ϕ, ψ, χ with respect to s, u, v is different from zero.

If P_0 is a point (x_0, y_0, z_0) on $\mathfrak{L}$ corresponding to the value u_0, then for any $\eta>0$ a value $\rho>0$ can be selected so that $|s(x, y, z)|<\eta$ whenever (x, y, z) is in the ρ-neighborhood of P_0. For the values (s, u, v) in $\mathfrak{S}'_\epsilon$ for which $s\geqq\eta$ define a closed set of points (x, y, z) distinct from P_0, whose minimum distance from the point P_0 is a positive constant $\rho\sqrt{3}$. Hence any (x, y, z) in the ρ-neighborhood of P_0 will have its distance from P_0 less than ρ, and will necessarily correspond to a value of s for which $0<s<\eta$. In a similar manner the points of $\mathfrak{S}_\epsilon$ for which $|u-u_0|\geqq\eta$ define a closed set of (x, y, z)-points distinct from P_0, and the continuity of $u(x, y, z)$ as stated in 2) can be proved.

It remains to consider the property 4). An application of Taylor's formula shows that the equations

$$\begin{aligned}
&\frac{1}{t}\{\phi(\sigma t,\ u_0+\tau t,\ v)-x_0\}=\frac{f(t)-x_0}{t},\\
&\frac{1}{t}\{\psi(\sigma t,\ u_0+\tau t,\ v)-y_0\}=\frac{g(t)-y_0}{t},\\
&\frac{1}{t}\{\chi(\sigma t,\ u_0+\tau t,\ v)-z_0\}=\frac{h(t)-z_0}{t},
\end{aligned}\tag{15}$$

reduce for $t=0$ to

$$\sigma\phi_s+\tau\alpha=f'(0), \qquad \sigma\psi_s+\tau\beta=g'(0), \qquad \sigma\chi_s+\tau\gamma=h'(0),\tag{16}$$

where $\sigma\geqq 0$ and the arguments of ϕ_s, ψ_s, χ_s are $(0, u_0, v)$. As a result of the statements proved on page 334, there can be but one value v_0 (modulo 2π) for which these equations are satisfied, and the corresponding direction (14) has the properties described above. It follows that equations (15) have for $t=0$ unique solutions

$$\sigma_0=\sin\omega\csc\theta, \qquad \tau_0=\cos\omega-\sin\omega\cot\theta, \qquad v_0$$

determined by equations (16). The functional determinant of equations (15) for the values $(\sigma_0, \tau_0, v_0, t = 0)$ is found with the help of the formula (10) to be the determinant (6). Hence these equations have continuous solutions $\sigma(t)$, $\tau(t)$, $v(t)$ reducing to σ_0, τ_0, v_0 for $t = 0$. The corresponding functions

$$s = \sigma t, \qquad u = u_0 + \tau t, \qquad \cos v = a(t), \qquad \sin v = b(t)$$

define solutions of equations (E), and hence must coincide with the functions $s(t)$, $u(t)$, $a(t)$, $b(t)$ determined under 4). It follows readily that $s'(0)$, $u'(0)$ exist and have the values σ_0, τ_0, while $a(t)$, $b(t)$, approach $\cos v_0$ and $\sin v_0$, respectively, as t approaches zero.

GROUPS GENERATED BY TWO OPERATORS (s_1, s_2) SATISFYING THE EQUATION $s_1 s_2^2 = s_2 s_1^2$.*

BY

G. A. MILLER

§ 1. *Introduction.*

In 1878 CAYLEY considered the groups generated by two operators (s_1, s_2) satisfying the equation $s_1 s_2 = s_2^2 s_1^2$ and observed the interesting theorem that it is not possible to represent all the operators of such a group in the form $s_1^\alpha s_2^\beta$ except in the special trivial case in which the group is cyclic.† This theorem is of historic interest as it is one of the earliest theorems relating to a general category of abstract groups. More recently NETTO published a few additional general results which may be deduced from this equation and he also determined the possible groups when the orders of s_1, s_2 are both less than 6.‡ The results obtained by NETTO were extended by the author of the present paper, mainly by means of special considerations, as regards the possible groups in which the two generating operators are of order 6.§

For convenience the equation under consideration is written, in the present paper, in the form $s_1 s_2^2 = s_2 s_1^2$, and a number of new general results are deduced from it. By means of these the known results are obtained much more easily than in the earlier papers. It is also proved that the two equations $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ imply that $s_2^5 = 1$ and hence either the first or the second of the three generational relations $s_1^5 = 1$, $s_2^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ given by NETTO in the article cited above is redundant. This is of interest since it proves that the non-cyclic group of order 55 is the only non-abelian group which can be generated by two operators satisfying the two conditions $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ and hence it establishes contact between the present paper and the one devoted to the "Finite groups which may be defined by two operators satisfying two conditions."‖

Among the other results the following are perhaps of most interest. There

* Presented to the Society, December 29, 1909.

† CAYLEY, Messenger of Mathematics, vol. 7 (1878), p. 188.

‡ NETTO, Crelle's Journal, vol. 128 (1905), p. 243.

§ Quarterly Journal of Mathematics, vol. 40 (1909), p. 197.

‖ American Journal of Mathematics, vol. 31 (1909), p. 167.

is an infinite system of solvable groups each of which may be generated by two operators which satisfy the three conditions $s_1^6 = 1$, $s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$; but there are only two groups which may be generated by two operators satisfying the three conditions $s_1^7 = 1$, $s_2^7 = 1$, $s_1 s_2^2 = s_2 s_1^2$. Combining these results with those known earlier we may say that the only groups which can be generated by two operators satisfying the three conditions,

$$s_1^\alpha = 1, \qquad s_2^\alpha = 1, \qquad s_1 s_2^2 = s_2 s_1^2,$$

are: The group of order 2 when $\alpha = 2$; the group of order 3 and the tetrahedral group when $\alpha = 3$; the cyclic groups of orders 2 and 4, and the holomorph of the group of order 5 when $\alpha = 4$; the group of order 5 and the non-cyclic group of order 55 when $\alpha = 5$; the cyclic groups of orders 2, 3 and 6, and an infinite system of solvable non-abelian groups when $\alpha = 6$; the group of order 7 and the non-cyclic group of order 203 when $\alpha = 7$. The fact that for $\alpha = 6$ there is an infinite system of possible groups, while there are only two for the larger value 7 of α, is to be emphasized.

§ 2. *General considerations.*

By writing the equation $s_1 s_2^2 = s_2 s_1^2$ in the form $s_1 s_2^2 s_1^{-1} = s_2 s_1$, and raising both members to the same power, there results the formula

$$s_1 s_2^{2n} s_1^{-1} = (s_2 s_1)^n.$$

In a similar manner we obtain the formula

$$s_2 s_1^{2n} s_2^{-1} = (s_1 s_2)^n.$$

From these two formulas we see directly that the three operators s_1^2, s_2^2, $s_1 s_2$ are of the same order, and that either s_1, s_2 have the same order or else the order of one of these operators is an odd number and the order of the other is twice this odd number. These results may be expressed in the form of a theorem as follows: *If two operators satisfy the equation $s_1 s_2^2 = s_2 s_1^2$, their squares are of the same order as their product.*

By transforming the given formulas with respect to s_1^{-1}, s_2^{-1}, respectively, we obtain

$$s_1^2 s_2^{2n} s_1^{-2} = (s_1 s_2)^n \qquad s_2^2 s_2^{2n} s_1^{-2} = (s_2 s_1)^n,$$

and hence

$$(A) \qquad s_1 s_2^{2n} s_1^{-1} s_2^2 s_1^{-2n} s_2^{-2} = s_2 s_1^{2n} s_2^{-1} s_1^2 s_2^{-2n} s_1^{-2} = 1.$$

The commutator of s_1, s_2 is $s_1^{-1} s_2^{-1} s_1 s_2 = s_1 \cdot s_1^{-2} s_2^{-1} \cdot s_1 s_2 = s_1 s_2^{-1}$, and hence

$$(B) \qquad (s_1^{-1} s_2^{-1} s_1 s_2)^2 = (s_1 s_2^{-1})^2 = s_1^3 \cdot s_1^{-2} s_2^{-1} \cdot s_1 s_2^{-1} = s_1^3 s_2^{-3}.$$

The group generated by the two commutators $s_1 s_2^{-1}$, $s_2^{-1} s_1$ is invariant under

the group G generated by s_1, s_2 since

$$s_1^{-1} s_2^{-1} s_1^2 = s_1 \cdot s_1^{-2} s_2^{-1} s_1^2 = s_1 s_2^{-2} s_1 = s_1 s_2^{-1} \cdot s_2^{-1} s_1,$$

$$s_2^{-1} \cdot s_2^{-1} s_1 \cdot s_2 = s_2^{-1} s_1^2 s_2^{-1} = s_2^{-1} s_1 \cdot s_1 s_2^{-1}.$$

As the quotient group of G with respect to this group is cyclic it results that this invariant subgroup is the commutator subgroup of G. That is, *if two operators satisfy the equation $s_1 s_1^2 = s_2 s_1^2$, they generate a group whose commutator subgroup is generated by two conjugate commutators and whose commutator quotient is cyclic.*

The commutator $s_1 s_2^{-1}$ is transformed into $s_2^{-1} s_1$ by each of the two operators s_1, s_2. It is also transformed into the same operator by each of the two operators s_1^3, s_2^3, since $(s_1 s_2^{-1})^2 = s_1^3 s_2^{-3}$. The two operators $s_1 s_2 s_1$, $s_2 s_1 s_2$ must have the same order since they may be obtained by transforming $s_2 s_1^2$, $s_1 s_2^2$ by s_1^{-1}, s_2^{-1} respectively. Hence $s_2^{-2} s_1 s_2 s_1 s_2^2$, $s_1^{-2} s_2 s_1 s_2 s_1^2$ must also have the same order. The last two operators are equal respectively to $s_2^{-1} s_1^4$, $s_1^{-1} s_2^4$, as may be seen by employing the relation $s_1 s_2^2 = s_2 s_1^2$. This fact was proved by Netto by means of more lengthy considerations. For convenience in the following applications the equation $s_1 s_2^2 = s_2 s_1^2$ is written also in the following forms:

$$s_2^{-2} s_1^{-1} = s_1^{-2} s_2^{-1}, \qquad s_1^{-1} s_2 = s_2^2 s_1^{-2}, \qquad s_2^{-1} s_1 = s_1^2 s_2^{-2}.$$

Throughout the following two paragraphs it will be assumed that s_1 and s_2 are both of odd order. Hence formula (A) may be written in the simpler form

$$(A') \qquad s_1 s_2^m s_1^{-1} s_2^2 s_1^{-m} s_2^{-2} = s_2 s_1^m s_2^{-1} s_1^2 s_2^{-m} s_1^{-2} = 1,$$

where m is an arbitrary integer. If we let $m = 1$ there results

$$(A'') \qquad s_1 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2^{-2} = s_1 s_2 s_1^{-2} s_2 s_1 s_2^{-2} = 1.$$

From the last equation we have

$$s_2 (s_1 s_2^{-1})^m s_2^{-1} = s_1 (s_1 s_2^{-1})^m s_1^{-1}.$$

Hence $s_1 s_2^{-1}$, $s_2^{-1} s_1$ are commutative. That is, *if two operators of odd order satisfy the equation $s_1 s_2^2 = s_2 s_1^2$ they generate a solvable group whose commutator subgroup is either cyclic or the direct product of two cyclic groups.*

All the exponents of the second member of (A'') may be multiplied by 2 without changing its value, since $s_1 s_2 = s_1^2 s_2^2 s_1^{-2}$ and $s_2 s_1 = s_2^2 s_1^2 s_2^{-2}$. Hence the equation

$$s_1^2 s_2^2 s_1^{-4} s_2^2 s_1^2 s_2^{-4} = 1.$$

By employing the given relations between s_1, s_2 and especially those which can be directly obtained from $s_1 s_2^n s_1^{-1} = s_2^2 s_1^n s_2^{-2}$, $s_2 s_1^n s_2^{-1} = s_1^2 s_2^n s_1^{-2}$, it is not difficult

to verify the following equations:

$$(s_1 s_2^{-1})^5 = s_1^3 s_2^{-3} \cdot s_1 s_2^{-1} \cdot s_1^3 s_2^{-3} = s_1^3 s_2^{-4} s_1^2 s_2 s_1 s_2^{-3} = s_1^3 s_2^{-4} s_1^3 s_2^2 s_1^{-1} s_2^{-3}$$
$$= s_1^3 s_2^{-4} s_1^3 s_2^4 s_1^{-2} s_2^{-4} = s_1^3 s_2^{-4} s_1 s_2 s_1^4 s_2^{-5} = s_1^3 s_2^{-3} s_1^2 s_2^{-1} s_1^4 s_2^{-5}$$
$$= s_1^3 s_2^{-2} s_1^{-1} s_2^{-1} s_1^6 s_2^{-5} = s_1 s_2^{-2} s_1^6 s_2^{-5}.$$

Hence it results directly that

$$(B) \qquad (s_1 s_2^{-1})^{4n} = s_2^{-1} (s_1^6 s_2^{-6})^n s_2.$$

From the equations

$$1 = s_2 s_1 s_2^{-1} s_1^2 s_2^{-1} s_1^{-2} = s_2 s_1 s_2^{-2} s_1 s_2 s_1^{-2} = s_2 s_1 s_2^{-1} s_1^2 s_2^{-1} s_1^{-2} = s_2 s_1^3 s_2^{-2} s_1^3 s_2^{-2} s_1^{-3},$$

it follows that $s_2^2 \cdot s_2^{-1} s_1^3 \cdot s_2^{-2} = s_1^3 s_2^2 s_1^{-3}$, and hence the two operators $s_2^{-1} s_1^3$, s_2^2 have the same order. In a similar manner we observe from the equations

$$1 = s_1 s_2^{-1} s_1^{-1} s_2^2 s_1 s_2^{-2} = s_2^{-2} s_1 s_2^{-1} s_1^{-1} s_2^2 s_1 = s_2^{-2} s_1 s_2^{-1} s_1^{-2} s_2 s_1^3$$
$$= s_2^{-2} s_1^3 s_2^{-2} s_1^{-3} s_2 s_1^3 = s_2^2 \cdot s_2^{-4} s_1^3 \cdot s_2^{-2} \cdot s_1^{-3} s_2 s_1^3$$

that $s_2^{-4} s_1^3$ is of the same order as s_2. That is, *when s_1, s_2 are both of odd order all of the following five operators: s_1, s_2, $s_1 s_2$, $s_2^{-1} s_1^3$, $s_2^{-4} s_1^3$ are of the same order.*

§ 3. *Several Special Cases.*

If we add to the equation $s_1 s_2^2 = s_2 s_1^2$ the additional equation $s_1^n = 1$, it is not difficult to determine all the possible groups for the different values of n from 2 to 5. If $n = 2$ it results directly from the former of these equations that $s_1 = s_2^{-1}$ and hence the group (G) generated by s_1, s_2 is of order 2. If $n = 3$ it results from the preceding section that the order of s_2 is either 3 or 6 and that the order of the commutator $s_1 s_2^{-1}$ is either 2 or 4, since $(s_1 s_2^{-1})^2 = s_1^3 s^{-3}$. If s_2 is of order 3 the order of $s_1 s_2^{-1}$ is 2 and hence G is generated by two operators of order 3 whose product is of order 2. It is well known that every pair of operators which satisfy these conditions generate the tetrahedral group. On the other hand, if s_2 is of order 6 when $n = 3$, the order of $s_1 s_2^{-1}$ is 4 and $(s_1 s_2^{-1})^2 = s_2^3$. It is known that the group of order 24 which does not involve a subgroup of order 12 is the only group that can be generated by two operators which fulfil these conditions.*

When $n = 4$ it results from the preceding section that s_2 is also of order 4 and that $s_1 s_2$ is of order 2. Hence, $s_2 s_1 = s_1^{-1} s_2^{-1}$, and $s_1^{-2} \cdot s_1 s_2^{-1} \cdot s_1^2 = s_1^{-1} s_2^{-1} \cdot s_1^2 = s_2 s_1^3 = s_2 s_1^{-1} = (s_1 s_2^{-1})^{-1}$. On the other hand, $s_1^{-1} \cdot s_1 s_2^{-1} \cdot s_1 = s_2^{-1} s_1 = s_2^3 s_1^{-3} = (s_1 s_2^{-1})^{-2}$. That is, s_1^2 transforms $s_1 s_2^{-1}$ both into its 4th power and also into its

*Transactions of the American Mathematical Society, vol. 8 (1907), p. 4. An operator is said to fulfil the condition $s_1^n = 1$ if it is of order n; it is said to satisfy this condition if its order is a divisor of n.

inverse, and hence the order of $s_1 s_2^{-1}$ is 5. Since s_1 transforms $s_1 s_2^{-1}$ into the inverse of its square, it has been proved that *G is the holomorph of the group of order* 5 *when* $n = 4$. From the preceding section it results that the order of s_2 is either 5 or 10 when $n = 5$. We shall now prove that it cannot be 10. This will be done by assuming that s_1, s_2 fulfil the conditions $s_1^5 = 1$, $s_2^{10} = 1$, $s_1 s_2^2 = s_2 s_1^2$ and by showing that this assumption leads to a contradiction.

Suppose that s_1, s_2 fulfill the three conditions

$$s_1^5 = 1, \qquad s_2^{10} = 1, \qquad s_1 s_2^2 = s_2 s_1^2.$$

From formula (A) of the preceding section we obtain

$$s_1 s_2^6 s_1^{-1} s_2^2 s_1^{-1} s_2^{-2} = 1.$$

Hence

$$s_1 s_2^5 \cdot s_2 s_1^2 \cdot s_1^2 s_2^2 s_1^{-1} \cdot s_2^{-2} = s_1 s_2^5 s_1 s_2^2 s_1 s_2 s_1 s_2^{-2} = s_1 s_2^5 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^{-4}$$

$$= s_1 s_2^5 s_1 s_2^3 s_1^{-1} s_2^{-4} = s_1 s_2^5 s_1 s_2^5 s_1^{-2} s_2^5 = 1.$$

From the last of these equations it is evident that the conjugates of s_2^5 with respect to s_1^{-1}, s_2^{-2} are commutative, since the product of two operators of order 2 $(s_1 s_2^5 s_1^{-1}, s_1^2 s_2^5 s_1^{-2})$ is of order 2. Transforming this last equation by s_1^{-1} leads to the equation

$$s_1^3 s_2^5 s_1^{-3} = s_1 s_2^5 s_1^{-1} \cdot s_1^2 s_2^5 s_1^{-2} = s_2^5.$$

Since s_1^3 is commutative with s_2^5 and since s_1^3 generates s_1 it results that s_1 is commutative with s_2^5, and hence we have $s_2^{10} = s_2^5 = 1$. As this is contrary to the hypothesis, it has been proved that *the two equations* $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ *imply that* $s_2^5 = 1$.

We proceed to determine all the groups which can be generated by two operators satisfying the two conditions

$$s_1^5 = 1, \qquad s_1 s_2^2 = s_2 s_1^2.$$

As $s_1^{-3} \cdot s_1 s_2^{-1} \cdot s_1^3 = s_1^{-2} s_2^{-1} s_1^3 = s_2^{-2} s_1^2 = s_2^3 s_1^{-3} = (s_1 s_2^{-1})^{-2}$ according to formula (B) of the preceding section, it results that $s_1^{-15}(s_1 s_2^{-1}) s_1^{15} = (s_1 s_2^{-1})^{-32}$ and hence the order of $s_1 s_2^{-1}$ must divide 33. To prove that this order is 11 or 1 we may proceed as follows: $s_1^{-6} \cdot s_1 s_2^{-1} \cdot s_1^6 = s_1^{-1} s_1 s_2^{-1} s_1 = s_2^{-1} s_1 = (s_1 s_2^{-1})^4$. Multiplying both members of the last equation by $s_1^3 s_2^{-3} = (s_1 s_2^{-1})^2$, we find that $(s_1 s_2^{-1})^6 = s_1^3 s_2 s_1$; hence $(s_1 s_2^{-1})^{11} = s_2 s_1^{-1} \cdot s_1^3 s_2 s_1^{-1} s_2 s_1 = s_2 s_1^2 s_2 s_1^{-1} s_2 s_1 = s_1 s_2^3 s_1^{-1} s_2 s_1$ $= s_1 \cdot s_2^{-2} s_1^{-1} \cdot s_2 s_1 = s_1 \cdot s_1^{-2} s_2^{-1} \cdot s_2 s_1 = 1$. As $s_1 s_2^{-1}$ is of order 11 or 1 it results that G is either the non-cyclic group of order 55 or the group of order 5. That is, *the group of order* 5 *and the non-cyclic group of order* 55 *are the only two groups which can be generated by two operators which satisfy both of the conditions* $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$.

§4. *Groups whose Two Generators satisfy the Three Conditions* $s_1^6=1,\ s_2^6=1,\ s_1s_2^2=s_2s_1^2$.

In the preceding section we considered all the possible groups when the order of at least one of the operators s_1, s_2 is less than 6. Hence we shall assume throughout the present section that each of these operators is actually of order 6. We shall first prove that each of the two commutators $s_1s_2^{-1}$, $s_2^{-1}s_1$ which generate the commutator subgroup of G must transform the square of the other into its inverse. This interesting fact may be established as follows:

$$s_2^{-1}s_1\cdot s_1s_2^{-1}\cdot s_1^{-1}s_2=s_2^{-1}s_1^2s_2^{-1}\cdot s_1^{-1}s_2=s_2^{-1}s_1^2s_2s_1^{-2}.$$

Hence it results that

$$s_2^{-1}s_1(s_1s_2^{-1})^2s_1^{-1}s_2=s_2^{-1}s_1^2s_2\cdot s_1^{-2}s_2^{-1}\cdot s_1^2s_2s_1^{-2}=s_2^{-1}s_1^2\cdot s_2^{-1}s_1\cdot s_2s_1^{-2}$$
$$=s_2^{-1}\cdot s_1^{-2}s_2^{-1}\cdot s_1^{-2}=s_2^{-3}s_1^{-3}=(s_1s_2^{-1})^{-2}.$$

In a similar manner we may prove that $s_1s_2^{-1}$ transforms $s_2^{-1}s_1$ into its inverse, as follows:

$$s_1s_2^{-1}\cdot s_2^{-1}s_1\cdot s_2s_1^{-1}=s_1s_2^{-2}s_1s_2s_1^{-1}=s_1s_2^{-1}s_1^2s_2^{-1}s_1^{-1},$$
$$s_1s_2^{-1}(s_2^{-1}s_1)^2s_2s_1^{-1}=s_1s_2^{-1}\cdot s_1^2s_2^{-2}\cdot s_1^2s_2^{-1}s_1^{-1}=s_1s_2^{-2}s_1^3s_2^{-1}s_1^{-1}$$
$$=s_1^{-1}s_2^{-1}\cdot s_1^{-2}s_2^{-1}\cdot s_1^{-1}=s_1^{-1}s_2^{-3}s_1^{-2}=(s_2^{-1}s_1)^{-2}.$$

Having proved that the commutator subgroup of G is generated by two operators each of which transforms the square of the other into its inverse we proceed to consider this general category of groups. That is, we shall consider the category of groups generated by t_1, t_2 where no restrictions are imposed on t_1, t_2 except that they satisfy the two equations

$$t_1^{-1}t_2^2t_1=t_2^{-2},\qquad t_2^{-1}t_1^2t_2=t_1^{-2}.$$

These conditions imply that $(t_1t_2)^2=(t_2t_1)^{-2}$, and hence each of the three cyclic groups generated by the operators t_1^2, t_2^2, $(t_1t_2)^2$ respectively is invariant under the group K generated by t_1, t_2. Any two of these three cyclic subgroups have at most two common operators, since each one of the three operators t_1, t_2, t_1t_2 transforms into their inverses all the operators of two of these three cyclic groups and those of the third into themselves. That is, all the operators of the three groups $\{t_2^2,(t_1t_2)^2\}$, $\{t_1^2,(t_1t_2)^2\}$, $\{t_1^2,t_2^2\}$ are transformed into their inverses by t_1, t_2, t_1t_2 respectively.

From the preceding results it is evident that the abelian group $\{t_1^2,t_2^2,(t_1t_2)^2\}$ is invariant under K. Each of the three groups

$$\{t_1,t_2^2,(t_1t_2)^2\},\ \{t_1^2,t_2,(t_1t_2)^2\},\ \{t_1^2,t_2^2,t_1t_2\}$$

is also invariant under K since $t_2^{-1} t_1 t_2 = t_2^{-2} t_1^{-1} (t_1 t_2)^2$. As a consequence, the quotient group of K with respect to $\{t_1^2, t_2^2, (t_1 t_2)^2\}$ is either the four group or a subgroup of this group, and K is always solvable. Hence the theorem: *If each of two operators transforms the square of the other into its inverse, these operators must also transform the square of their product into its inverse, and these three squares generate an abelian subgroup which is invariant under the group generated by the two operators. The index of this subgroup is a divisor of* 4.

It is not difficult to construct such groups in which the orders of $t_1^2, t_2^2, (t_1 t_2)^2$ are three arbitrary numbers. To do this we may write, in distinct sets of letters, three cyclic substitution groups of the required orders, each generator being composed of two equal cycles when its order is even. We then select t_1, t_2 in such a way that they generate respectively the first two of these cyclic groups and that their components involving the letters of the other cyclic groups are of order 2, transform the operators of these cyclic groups into their inverses and give a product of the required order. It follows directly from the properties of the dihedral group that these substitutions can be so selected that t_1, t_2 have the given properties, and this establishes the existence of K for any arbitrary set of numbers for the orders of $t_1^2, t_2^2, (t_1 t_2)^2$.

Having established some fundamental properties of the general category of groups which may be generated by two operators, each of which transforms the square of the other into its inverse, we return to the operators under consideration. The three operators,

$$(s_1 s_2^{-1})^2 = s_1^3 s_2^{-3}, \qquad (s_2^{-1} s_1)^2 = s_1^2 s_2^{-3} s_1, \qquad (s_1 s_2^{-2} s_1)^2 = s_1 s_2^{-3} s_1^2,$$

generate an invariant subgroup under the commutator group of G and its index under this group divides 4 according to the general theory. It is easy to verify that these three operators together with their inverses constitute a complete set of conjugates under G and hence the group generated by them is also invariant under G.

These results suffice to prove that G can be constructed when the order of $(s_1 s_2^{-1})^2$ is an arbitrary number. The general method can be easily deduced from the following example. Suppose that the order of $(s_1 s_2^{-1})^2$ is 3. Write the three cyclic substitutions abc, def, ghi and select a substitution of order 3

$$t = adg \cdot beh \cdot cfi$$

which merely permutes the corresponding letters of these cycles. Find a substitution of order 2 which transforms each of the cyclic substitutions into its inverse and is commutative with t. In the present case we may select $ab \cdot de \cdot gh$. For s_1 take the continued product of the second cycle, this substitution of order 2, and t. In the present case

$$s_1 = ab \cdot ef \cdot gh \cdot t = aeicfh \cdot bdg.$$

For s_2 we take the continued product of the first cycle, the component of the given substitution of order 2 which involves the letters of this cycle, and t.

$$s_2 = bc \cdot t = adg \cdot bficeh .$$

It is easy to see that any two substitutions selected in the manner in which s_1, s_2 were chosen must be of order 6 and must also satisfy the equation $s_1 s_2^2 = s_2 s_1^2$. The three cyclic substitutions generate a group whose order is the cube of that of one of the cycles. This is the group generated by

$$(s_1 s_2^{-1})^2, \ (s_2^{-1} s_1)^2, \ (s_1 s_2^{-2} s_1)^2 .$$

This abelian invariant subgroup is of index 4 under the commutator subgroup and of index 24 under G. In the special case under consideration G is of order 648 and it is the largest group which can be generated by two operators which satisfy the four conditions:

$$s_1^6 = 1, \ s_2^6 = 1, \ (s_1 s_2^{-1})^6 = 1, \ s_1 s_2^2 = s_2 s_1^2 .^*$$

If the order of $(s_1 s_2^{-1})^2$ had been even we should have proceeded in the same way except that we should have selected cyclic substitutions whose order is twice that of $(s_1 s_2^{-1})^2$ in place of substitutions like *abc*, *def*, *ghi* whose order is equal to that of $(s_1 s_2^{-1})^2$. But the case where $(s_1 s_2^{-1})^2$ is of odd order is sufficient to prove the theorem: *There is an infinite number of distinct groups each of which is generated by two operators satisfying the three conditions* $s_1^6 = s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$. *All such groups are solvable. The commutator subgroup of each of them is generated by two conjugate operators each of which transforms the square of the other into its inverse, and the order of each of the groups generated by* s_1, s_2 *is a divisor of three times the cube of the order of the commutator of* s_1, s_2.

If the commutator $s_1 s_2^{-1}$ of s_1, s_2 were of odd order, $s_2^{-1} s_1$ would also be of odd order. This is clearly impossible since each of these operators transforms the square of the other into its inverse. That is, *if two operators satisfy the three conditions* $s_1^6 = s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$, *their commutator must be of even order.* This theorem was proved by means of prolix considerations in volume 40 of the *Quarterly Journal of Mathematics*, page 201. It may be remarked that the other results of that article relating to the sets of conditions under consideration follow almost immediately from the theorems of the present paper and may be used to illustrate these theorems.

§ 5. *Groups whose two generators satisfy either the three conditions* $s_1^7 = s_2^7 = 1$, $s_1 s_2^2 = s_2 s_1^2$; *or the conditions* $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_2 s_1^2$.

Throughout the first two paragraphs it will be assumed that the first one of these two sets of conditions is satisfied.

*Quarterly Journal of Mathematics, vol. 40 (1909), p. 203.

By making $n = 1$ in formula B' of section 2 there results $(s_1 s_2^{-1})^4 = s_2^{-1} s_1^{-1} s_2^2$. Hence $(s_1 s_2^{-2})^5 = s_1 s_2^{-2} s_1^{-1} s_2^2 = s_1^{-1} s_2 = (s_2^{-1} s_1)^{-1}$. That is, the group generated by $s_1 s_2^{-1}$ involves $s_2^{-1} s_1$ and is therefore invariant under G. Hence the commutator subgroup of G is the cyclic group generated by $s_1 s_2^{-1}$. Since $s_1^{-1} \cdot s_1 s_2^{-1} \cdot s_1 = s_2^{-1} s_1 = (s_1 s_2^{-1})^{-5}$ it results that $(-5)^7 \equiv 1$ modulo the order of $s_1 s_2^{-1}$. Hence the order of $s_1 s_2^{-1}$ divides $6 \cdot 29 \cdot 449$. To verify that this order is 29 we may proceed as follows:

$$(s_1 s_2^{-1})^{29} = (s_1 s_2^{-1})^4 (s_1^{-1} s_2)^5 = (s_1 s_2^{-1})^4 s_1^{-1} s_2 (s_1^{-1} s_2)^4 = (s_1 s_2^{-1})^4 s_1^{-1} s_2 s_1^{-1} (s_1 s_2^{-1})^{-4} s_1$$

$$= s_2^{-1} s_1^{-1} s_2^2 s_1^{-1} s_2 s_1^{-1} s_2^{-2} s_1 s_2 s_1 = s_2^{-1} s_1^{-1} s_2^4 s_1^{-3} s_2^{-2} s_1 s_2 s_1$$

$$= s_2^{-1} s_1^{-1} s_2^4 s_1^{-2} s_2^{-2} s_1^2 = s_2^{-1} s_1^{-1} s_2^2 s_1 s_2^{-2} s_1 = 1.$$

Since the order of $s_1 s_2^{-1}$ is 29 or 1, G must be the non-cyclic group of order 203 or the group of order 7. It is evident that G may be of order 7. That it may also be of order 203 may be proved in the following manner.

Suppose that t is an operator of order 19 and that $s_1^{-1} t s_1 = t^{24}$. Since 24 belongs to exponent 7 modulo 29 we may assume that s_1 is of order 7 and hence $s_1 t$ is also of order 7. The two operators $s_1, s_1 t$ evidently generate the non-cyclic group of order 203 and $s_1 (s_1 t)^2 = s_1^2 t s_1 t = s_1^3 s_1^{-1} t s_1 t = s_1^3 t^{25} = s_1 t s_1^2$ since $24^2 \equiv 25 \bmod 29$. These results prove that there is only one prime number p which satisfies the condition that it is possible to find a number α belonging to exponent 7 and satisfying the equation $\alpha + 1 \equiv \alpha^2 \bmod p$. This prime number is 29 and α is 24. This result may also be stated as follows: *If α belongs to exponent 7 with respect to a prime number p and if $\alpha + 1 \equiv \alpha^2$ mod p then $p = 29$ and $\alpha = 25$.* We are thus led to a characteristic property of the prime 29. Similar characteristic properties of the primes 5 and 11 may be deduced from the results of section 3, and in what follows we shall establish also such a characteristic property of 19.

When $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_2 s_1^2$ formula B' gives rise to the equations

$$(s_1 s_2^{-1})^{4n} = s_2^{-1} (s_1^{-3} s_2^3)^n s_2 = s_2^{-1} s_1^{-3} (s_1 s_2^{-1})^{-2n} s_1^3 s_2.$$

Since $s_1^3 s_2$ transforms $(s_1 s_2^{-1})^2$ into a power of itself we shall first prove that the order of $s_1^3 s_2$ is 9. This fact is established by the following equations:

$$(s_1^3 s_2)^4 = s_1^3 s_2 s_1^3 s_2 s_1^3 s_2 s_1^3 s_2 = s_1^4 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2$$

$$= s_1^4 s_2^3 s_1^2 s_2^{-1} s_1^4 s_2^2 s_1 s_2 = s_1^4 s_2^4 s_1^{-1} s_2^{-1} s_1^6 s_2^2 s_1 s_2$$

$$= s_1^4 s_2^2 s_1^{-2} s_2^{-2} s_1^6 s_2^2 s_1 s_2 = s_1^4 s_2^6 s_1^{-4} s_2^{-1} s_1^{-2} s_2^2 s_1 s_2$$

$$= s_1^4 s_2^6 s_1^{-6} s_2 s_1^{-1} s_2 s_1 s_2 = s_1^4 s_2^6 s_1^3 s_2^3 s_1^{-1} s_2.$$

Hence $(s_1^3 s_2)^3 = s_1^4 s_2^6 s_1^3 s_2^3 s_1^{-4} = s_1^4 s_2^6 \cdot s_1^3 \cdot (s_1^4 s_2^6)^{-1}$.

Since s_1^3 is of order 3 the order of $s_1^3 s_2$ must be 9, and from the fact that this operator transforms $(s_1 s_2^{-1})^2$ into $(s_1 s_2^{-1})^{-4}$ it results that its 9th power transforms $(s_1 s_2^{-1})^2$ into its -512th power. That is, the order of $s_1 s_2^{-1}$ must divide $1026 = 2 \cdot 19 \cdot 27$. In view of this result we are led to inquire whether the order of $s_1 s_2^{-1}$ could be 19. That this is possible results directly from the fact that if t is an operator of order 19, and if $s_1^{-1} t s_1 = t^5$ then s_1, $s_1 t$ satisfy the equation $s_1 (s_1 t)^2 = s_1 t s_1^2$, as $6 \equiv 25 \mod 19$. It is also easy to see that the group of order 36 generated by $s_1 = abc \cdot t_1$, $s_2 = bdc \cdot t_1$, where t_1 is invariant and of order 9, fulfils the conditions imposed upon the generators at the beginning of this paragraph. Hence it has been proved that *there is more than one non-abelian group which involves two generators satisfying the conditions* $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_1 s_2^2$, *but the only prime numbers which divide the order of such a group are* 2, 3, *and* 19.

CONGRUENCES OF THE ELLIPTIC TYPE*

BY

LUTHER PFAHLER EISENHART

Introduction.

Let L be a line of a rectilinear congruence; X, Y, Z, its direction-cosines; and x, y, z, the coördinates of the point in which L cuts a surface of reference S. These six quantities are functions of two parameters, say u and v, which we assume to be real. As usual we put

$$E = \Sigma\left(\frac{\partial X}{\partial u}\right)^2, \qquad F = \Sigma\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}, \qquad G = \Sigma\left(\frac{\partial X}{\partial v}\right)^2,$$

$$e = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial u}, \quad f = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial u}, \quad f' = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial v}, \quad g = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial v}. \tag{1}$$

Of all the ruled surfaces, formed by lines of the congruence, which pass through L two at most are developable. They are defined by the equation

$$(Ef' - Fe)du^2 + (Eg + Ff' - Ff - Ge)du\,dv + (Fg - Gf)dv^2 = 0. \tag{2}$$

In the case of normal congruences, congruences of Guichard, cyclic congruences and congruences of tangents to a real family of curves on a surface, the integrals of equation (2) are real. But there is a large variety of congruences for which the integrals of this equation are imaginary. We say that a congruence is of the *hyperbolic* or *elliptic type* according as the two values of dv/du given by (2) are real or imaginary. This paper deals with congruences of the latter type, and particularly with pairs of ruled surfaces which are real only in this case and which possess properties analogous to those of the developable surfaces of a congruence of the hyperbolic type.

One of these systems may be defined analytically by means of the following theorem of CIFARELLI:†

Given two quadratic differential forms

$$a_1du^2 + 2a_2du\,dv + a_3dv^2, \qquad b_1du^2 + 2b_2du\,dv + b_3dv^2, \tag{3}$$

of which the first is definite, that is, $a_1a_3 - a_2^2 > 0$; if one forms the Jacobian

* Presented to the Society September 13, 1909.

† *Le Congruenze*, Annali di Matematica, ser. 3, vol. 2 (1899), p. 148.

of these forms and equates to zero the Jacobian of the resulting quadratic form and of the first of (3), the solutions of the resulting differential equation define a real transformation, say $u' = \phi(u, v)$, $v' = \psi(u, v)$, which changes the forms (3) into two

$$a_1' du'^2 + 2a_2' du'dv' + a_3' dv'^2, \qquad b_1' du'^2 + 2b_2' du'dv' + b_3' dv'^2,$$

which are such that

$$\frac{a_1'}{b_1'} = \frac{a_3'}{b_3'}, \qquad a_2' = 0. \tag{4}$$

Since the left-hand member of equation (2) is a definite quadratic form, it may be taken as the first of (3). If we take for the second the square of the linear element of the spherical representation, namely

$$d\sigma^2 = Edu^2 + 2Fdu\,dv + Gdv^2, \tag{5}$$

the parametric ruled surfaces $u' = \text{const.}$, $v' = \text{const.}$ constitute a system which we shall study in detail. We call them the *characteristic ruled surfaces* of the congruence.

In § 1 the equation of the characteristic ruled surfaces is derived, and therefrom we discover properties of the lines of striction of these surfaces, of their spherical representation, and the fact that their parameters of distribution are equal to one another and to the harmonic mean of the maximum and minimum values of the parameter of distribution of all the surfaces through the line.

The determination of a congruence with an assigned spherical representation of its characteristic ruled surfaces is investigated in § 2, and the results are applied in § 3 to the discussion of congruences whose characteristic surfaces meet the surface of reference in its asymptotic lines. These congruences are of the RIBAUCOUR type. The necessary and sufficient conditions for their existence are found, and an example of such congruences is given in § 4. Incidentally a theorem is derived concerning the case where the characteristic lines on a surface correspond to the asymptotic lines on an associate surface.

Since the quadratic form (2) is definite, there exist an infinity of real transformations of variables such that in terms of the new variables the expression (2) is of the form $\lambda(du'^2 + dv'^2)$. We say that such a parametric system is *isothermic*. Section 5 deals with the determination of congruences with an assigned spherical representation of an isothermic system of ruled surfaces. If ρ denotes the distance from the middle point of a line to one of its two conjugate purely imaginary focal points, the surfaces which are the loci of the points at the distances $i\rho$, $-i\rho$ from the middle point are of interest in this theory. We call them the *pseudofocal surfaces* of the congruence.

When the pseudofocal surfaces correspond with parallelism of tangent planes, as in § 6, the congruences are of the kind studied by LILIENTHAL. He took

three functions of $u + iv$, written

$$x_1 + ix_2 = f_1(u + iv), \qquad y_1 + iy_2 = f_2(u + iv), \qquad z_1 + iz_2 = f_3(u + iv),$$

and considered the congruence of lines joining corresponding points of the surfaces which are loci of the points (x_1, y_1, z_1), (x_2, y_2, z_2). We say that two such surfaces are *conjugate-potential*. These *congruences of Lilienthal* are of the RIBAUCOUR type, and the spherical representation of their imaginary developables is similar to that of congruences whose focal surfaces are curves.

Conjugate-potential surfaces are associate to one another. Section 7 deals with congruences which consist of lines joining corresponding points on associate surfaces. Certain congruences of RIBAUCOUR possess this property in an infinity of ways, and of this group are the congruences of LILIENTHAL.

§ 1. *Characteristic Ruled Surfaces.*

Given a congruence of the elliptic type expressed in terms of any parametric system and with an arbitrary surface of reference. If we put

$$(6) \quad A = \frac{Ef' - Fe}{H}, \qquad B = \frac{Eg + F(f' - f) - Ge}{2H}, \qquad C = \frac{Fg - Gf}{H},$$

where $H = \sqrt{EG - F^2}$, the equation of the developables (2) may be written

$$(7) \qquad A du^2 + 2B du dv + C dv^2 = 0.$$

If we apply the theorem of Cifarelli to the left-hand member of this equation and to the right-hand member of (5), we find that the differential equation of the characteristic surfaces is reducible to

$$(8) \quad \begin{aligned} &[A(AG - CE) - 2B(AF - BE)]\, du^2 + 2[B(AG + CE) \\ &\qquad - 2ACF]\, du dv + [2B(BG - CF) - C(AG - CE)]\, dv^2 = 0. \end{aligned}$$

In order that the characteristic ruled surfaces be parametric, we must have

$$(9) \quad \begin{aligned} (AG - CE)A - 2B(AF - BE) &= 0, \\ (AG - CE)C - 2B(BG - CF) &= 0, \end{aligned}$$

which equations are reducible to

$$(10) \qquad \frac{A}{E} = \frac{C}{G}, \qquad B = 0,$$

unless

$$(11) \qquad A(BG - CF) - C(AF - BE) = 0.$$

But in the latter case the middle term also of (8) vanishes, so that the characteristic ruled surfaces are indeterminate in this case. Hence equations (10), or

in other form

$$(f+f')-F\left(\frac{e}{E}+\frac{g}{G}\right)=0, \qquad F(f'-f)+Eg-Ge=0, \tag{12}$$

constitute a necessary and sufficient condition that the characteristic surfaces be parametric.

If the middle surface of the congruence is taken as the surface of reference, we have*

$$(f+f')F-EG\left(\frac{e}{E}+\frac{g}{G}\right)=0. \tag{13}$$

Combining this equation and the first of (12), we have

$$f+f'=0 \qquad \frac{e}{E}+\frac{g}{G}=0, \tag{14}$$

and the second of equations (12) may be given either of the forms

$$Eg+Ff'=0, \qquad Ff+Ge=0. \tag{15}$$

In order to interpret the second of equations (14), we recall that the abscissa r, measured from the surface of reference, of the point P on a line L where the line of striction of the ruled surface defined by a value of dv/du meets L is given by †

$$r=-\frac{edu^2+(f+f')dudv+gdv^2}{Edu^2+2Fdudv+Gdv^2}. \tag{16}$$

From this and (14) it follows that the values of r for the two characteristic surfaces through L differ only in sign. Hence we have

THEOREM 1. *The lines of striction of the two characteristic ruled surfaces through a line L of the congruence meet L in points equidistant from the middle point.*

We call these the *characteristic points*.

Before proceeding we return to the consideration of the particular case for which equation (11) holds as well as (9), that is when the characteristic ruled surfaces are indeterminate. These conditions are equivalent to

$$\frac{AG-CE}{2B}=\frac{AF-BE}{A}=\frac{BG-CF}{C}=\lambda, \tag{17}$$

where λ denotes the factor of proportionality to be determined. The condition that these equations be consistent is

$$\begin{vmatrix} G & 2\lambda & E \\ F-\lambda & E & 0 \\ 0 & -G & F+\lambda \end{vmatrix} = 2\lambda(EG-F^2+\lambda^2)=0.$$

* E., p. 401. A reference in this form is to my *Differential Geometry*, Boston, 1909.

† E., p. 395.

Since the quantities are real, we must have $\lambda = 0$, and consequently equations (17) are reducible to

$$\frac{A}{E} = \frac{B}{F} = \frac{C}{G}.$$

Hence the congruences sought are such that their developables are represented on the sphere by the minimal lines of the latter. This is a characteristic property of isotropic congruences.* Hence we have:

THEOREM 2. *The characteristic ruled surfaces of an elliptic congruence are real and determinate, except when the congruence is isotropic.*

Isotropic congruences will be excluded from the subsequent discussion.

Let $\bar{r}$ and R denote the abscissæ, measured from the middle point of a line L, of a limit point P and of the characteristic point C on the same side of the middle point. If ω denotes the angle which the tangent plane at C to the corresponding characteristic surface makes with the tangent plane to the principal surface at P, we have from Hamilton's equation †

$$R = \bar{r}(\cos^2\omega - \sin^2\omega).$$

In like manner for the other characteristic ruled surface we have

$$R = -\bar{r}(\cos^2\omega' - \sin^2\omega').$$

From these equations we obtain

$$\cos 2\omega + \cos 2\omega' = 0. \tag{18}$$

Consequently the curves on the sphere which represent the characteristic ruled surfaces either form an orthogonal system, or they are equally inclined to the curves which bisect the angles between the curves representing the principal ruled surfaces.

In the former case, when these surfaces are parametric, we should have $F = 0$ and also $f + f' = 0$, from the first of (12). Moreover, from the second of (12) we should have $Eg = Ge$; that is, the congruence is isotropic. Hence $F \neq 0$, and we have

THEOREM 3. *The curves on the sphere which represent the characteristic ruled surfaces are equally inclined to the curves which bisect the angles between the images of the principal surfaces.*

The developables of a congruence of the hyperbolic type possess the same property.

By means of the preceding results we obtain the quadratic equation which the abscissæ of the characteristic points satisfy, when the parameters and surface of reference are any whatsoever. If R_1 and R_2 denote these abscisssæ

*E. p. 412.

†E, p. 397.

and r_1, r_2, the abscissæ of the limit points, we have from Hamilton's equation

$$R_1 = r_1 \cos^2\omega + r_2 \sin^2\omega,$$

$$R_2 = r_1 \cos^2\omega' + r_2 \sin^2\omega',$$

and from (18)

$$\cos^2\omega' = \sin^2\omega, \qquad \sin^2\omega' = \cos^2\omega.$$

From these equations we deduce

$$R_1 + R_2 = r_1 + r_2,$$

$$R_1 R_2 = (r_1 + r_2)^2 \frac{\cos^2\theta_0}{4} + r_1 r_2 \sin^2\theta_0,$$

where θ_0 denotes the angle between the spherical images of the characteristic surfaces. If the differential equation of the latter surfaces be written in the form

$$Ldu^2 + 2Mdudv + Ndv^2 = 0,$$

it is readily shown that

$$\cos^2\theta_0, \sin^2\theta_0 = \frac{(EN-2FM+GL)^2, \quad 4H^2(M^2-NL)}{E^2N^2-4EFMN+2EG(2M^2-NL)-4FGLM+4F^2LN+G^2L^2}.$$

Since moreover

$$r_1 + r_2 = \frac{(f+f')F - gE - eG}{EG - F^2}, \qquad r_1 r_2 = \frac{4eg - (f+f')^2}{EG - F^2},$$

we can readily determine the expressions of the coefficients of the quadratic equation satisfied by the abscissæ of the characteristic points. On account of the involved form of this equation we will not give it here.

We shall obtain another property of the characteristic ruled surfaces by recalling that the parameter of distribution of the ruled surface determined by a value of dv/du is given by*

$$p = \frac{Adu^2 + 2Bdudv + Cdv^2}{Edu^2 + 2Fdudv + Gdv^2}. \tag{19}$$

The first of equations (10) expresses the fact that the parameters of distribution of the two characteristic surfaces through a line are equal.

The differential equation (8) can be written

$$\frac{AG + CE - 2BF}{2(AC - B^2)} = \frac{Edu^2 + 2Fdudv + Gdv^2}{Adu^2 + 2Bdudv + Cdv^2}. \tag{20}$$

If now p_1 and p_2 denote the maximum and minimum values of p for all the ruled surfaces through a line, and if p_c denotes the value for the characteristic

* E, p. 424.

surfaces, equation (20) may be written, in consequence of (19),

$$\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)=\frac{1}{p_c}.$$

Hence we have

THEOREM 4. *The parameters of distribution of the two characteristic ruled surfaces through a line have the same value, namely the harmonic mean of the maximum and minimum values of the parameter for all the ruled surfaces through the line.*

The first part of this theorem is a consequence of the following theorem which can be readily proved:

THEOREM 5. *If the lines of striction of two ruled surfaces through a line L meet the latter at points equidistant from its middle point, the parameters of distribution of the two surfaces have the same value; and conversely.*

§ 2. *Spherical Representation of Characteristic Ruled Surfaces.*

It can be shown that the first derivatives of the coefficients of the surface of reference of a congruence are expressible in the form *

$$\begin{aligned}\frac{\partial x}{\partial u}&=\frac{eG-fF}{H^2}\frac{\partial X}{\partial u}+\frac{fE-eF}{H^2}\frac{\partial X}{\partial v}+\gamma X,\\ \frac{\partial x}{\partial v}&=\frac{fG-gF}{H^2}\frac{\partial X}{\partial u}+\frac{gE-fF}{H^2}\frac{\partial X}{\partial v}+\gamma_1 X,\end{aligned}\tag{21}$$

and similar equations in y and z, where γ and γ_1 are functions which must satisfy the conditions

$$\begin{aligned}&\frac{\partial e}{\partial v}-\frac{\partial f}{\partial u}-\left\{\begin{matrix}12\\1\end{matrix}\right\}'e+\left\{\begin{matrix}11\\1\end{matrix}\right\}'f-\left\{\begin{matrix}12\\2\end{matrix}\right\}'f'+\left\{\begin{matrix}11\\2\end{matrix}\right\}'g+F\gamma-E\gamma_1=0,\\ &\frac{\partial f'}{\partial v}-\frac{\partial g}{\partial u}-\left\{\begin{matrix}22\\1\end{matrix}\right\}'e+\left\{\begin{matrix}12\\1\end{matrix}\right\}'f-\left\{\begin{matrix}22\\2\end{matrix}\right\}'f'+\left\{\begin{matrix}12\\2\end{matrix}\right\}'g+G\gamma-F\gamma_1=0,\\ &\frac{\partial \gamma}{\partial v}-\frac{\partial \gamma_1}{\partial u}+f-f'=0,\end{aligned}\tag{22}$$

the Christoffel symbols being $\left\{\begin{matrix}rs\\t\end{matrix}\right\}'$ are formed with respect to the quadratic form (5).

We consider now a congruence of the elliptic type referred to its characteristic surfaces. We assume that the middle surface is the surface of reference and that R is the abscissa of the characteristic point of the surface $u=$ const. From (14), (15) and (16) we have

$$R=\frac{e}{E}=\frac{-g}{G}=-\frac{Ff}{EG}=\frac{Ff'}{EG}.\tag{23}$$

* E, pp. 406, 407.

The abscissæ of the limit and focal points are given by the respective equations*

$$\bar{r}^2 = \frac{(f+f')^2 - 4eg}{4H^2}, \qquad \rho^2 = \frac{ff' - eg}{H^2}. \tag{24}$$

From equations (23) and (24) we obtain

$$R = \bar{r} \sin \theta_0, \qquad R = i\rho \cos \theta_0, \tag{25}$$

where θ_0 denotes the angle between the central planes of the two characteristic surfaces.

In consequence of (23) equations (21) are reducible to

$$\frac{\partial x}{\partial u} = \frac{ER}{F}\frac{\partial X}{\partial v} + \gamma X, \qquad \frac{\partial x}{\partial v} = -\frac{GR}{F}\frac{\partial X}{\partial u} + \gamma_1 X. \tag{26}$$

From the first two of (22) we find that

$$\begin{aligned} \gamma &= -\frac{\partial}{\partial v}\left(\frac{ER}{F}\right) - \frac{R}{F}\left(E\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}' + G\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\right), \\ \gamma_1 &= \frac{\partial}{\partial u}\left(\frac{GR}{F}\right) + \frac{R}{F}\left(E\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}' + G\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}'\right), \end{aligned} \tag{27}$$

and the last of (22) may be written

$$\begin{aligned} &\frac{\partial^2}{\partial u^2}\left(\frac{GR}{F}\right) + \frac{\partial^2}{\partial v^2}\left(\frac{ER}{F}\right) + \frac{\partial}{\partial u}\left[\frac{R}{F}\left(E\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}' + G\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}'\right)\right] \\ &\qquad + \frac{\partial}{\partial v}\left[\frac{R}{F}\left(E\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}' + G\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\right)\right] + 2\frac{EG}{F}R = 0. \end{aligned} \tag{28}$$

Since equations (22) are sufficient conditions upon the functions e, f, f', g γ and γ_1 that equations (21) define the surface of reference of a congruence with a given spherical representation of linear element (5),† we have

THEOREM 6. *Given any system of curves on the sphere and let E, F, G denote the fundamental coefficients; each solution of equation (28) determines a congruence whose characteristic ruled surfaces are represented on the sphere by the given system of curves.*

From (26) we obtain by differentiation with respect to u and v the following

$$\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= \left(\frac{\partial}{\partial u}\log\frac{ER}{F} + \left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'\right)\frac{\partial x}{\partial u} - \left(\frac{E}{G}\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}' + \frac{F\gamma}{GR}\right)\frac{\partial x}{\partial v} + A_{11}X, \\ \frac{\partial^2 x}{\partial u \partial v} &= -\frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\frac{\partial x}{\partial u} - \frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\frac{\partial x}{\partial v} + A_{12}X, \\ \frac{\partial^2 x}{\partial v^2} &= \left(\frac{F\gamma_1}{ER} - \frac{G}{E}\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'\right)\frac{\partial x}{\partial u} + \left(\frac{\partial}{\partial v}\log\frac{GR}{F} + \left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}'\right)\frac{\partial x}{\partial v} + A_{22}X, \end{aligned} \tag{29}$$

* E., pp. 396, 399.

† E., p. 407.

where, for the sake of brevity, we have put

$$
\begin{aligned}
A_{11} &= \gamma\gamma_1 \frac{F}{GR} - \gamma\left(\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}' + \frac{\partial}{\partial u}\log\frac{ER}{F}\right) + \gamma_1\frac{E}{G}\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}' - ER + \frac{\partial\gamma}{\partial u},\\
A_{12} &= \frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\gamma_1 + \frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\gamma + \frac{\partial\gamma}{\partial v} - \frac{GE}{F}R,\\
&= \frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\gamma_1 + \frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\gamma + \frac{\partial\gamma_1}{\partial u} + \frac{GE}{F}R,\\
A_{22} &= -\gamma\gamma_1\frac{F}{ER} - \gamma_1\left(\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}' + \frac{\partial}{\partial v}\log\frac{GR}{F}\right) + \gamma\frac{G}{E}\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}' + GR + \frac{\partial\gamma_1}{\partial v}.
\end{aligned}
\tag{30}
$$

§ 3. *Congruences whose Characteristic Ruled Surfaces meet their Middle Surfaces in Asymptotic Lines.*

We consider in particular the case where the characteristic surfaces cut the middle surface in its asymptotic lines. From (29) and (30) it is seen that a necessary and sufficient condition for this is

$$A_{11} = A_{22} = 0.$$

In this case we have

$$A_{11}G + A_{22}E = G\frac{\partial\gamma}{\partial u} + E\frac{\partial\gamma_1}{\partial v} - \gamma G\frac{\partial}{\partial u}\log\frac{ER}{F} - \gamma_1 E\frac{\partial}{\partial v}\log\frac{GR}{F} = 0.$$

When the values of γ and γ_1 from (27) are substituted in this equation, the result is reducible to

$$\frac{\partial}{\partial u}\left[\frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}' + \left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}' + \frac{\partial}{\partial v}\log\sqrt{\frac{E}{G}}\right] = \frac{\partial}{\partial v}\left[\frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}' + \left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}' + \frac{\partial}{\partial u}\log\sqrt{\frac{G}{E}}\right]. \tag{31}$$

But this is a necessary and sufficient condition that the parametric curves on the sphere be the spherical representation of the characteristic conjugate system on a unique surface Σ.* The coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of Σ are given by the quadratures

$$\frac{\partial\bar{x}}{\partial u} = -\frac{\lambda}{G}\left(G\frac{\partial X}{\partial u} - F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial\bar{x}}{\partial v} = \frac{\lambda}{E}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right), \tag{32}$$

where λ is given by

$$
\begin{aligned}
\frac{\partial\log\lambda}{\partial u} &= \frac{\partial}{\partial u}\log\sqrt{\frac{E}{G}} - \left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\frac{E}{G} - \left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}',\\
\frac{\partial\log\lambda}{\partial v} &= \frac{\partial}{\partial v}\log\sqrt{\frac{G}{E}} - \left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\frac{G}{E} - \left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}'.
\end{aligned}
\tag{33}
$$

* Eisenhart, *Three particular systems of lines on a surface*, Transactions of the American Mathematical Society, vol. 5 (1904), p. 434.

From (26) and (32) we have

$$(34)\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial v}=0.$$

Hence S and Σ correspond with orthogonality of linear elements and consequently the congruence under consideration is of the Ribaucour type,* and Σ is the director-surface.

Referring to (29), we have that another necessary condition that the asymptotic lines on S be parametric is

$$\frac{\partial}{\partial v}\left(\frac{\partial}{\partial u}\log\frac{ER}{F}+\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'\right)=\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v}\log\frac{GR}{F}+\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}'\right).$$

This equation and (31) are equivalent to the two

$$(35)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\frac{G}{E}+\frac{\partial}{\partial u}\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}'-\frac{\partial}{\partial v}\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'=0,\qquad \frac{\partial}{\partial u}\frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'=\frac{\partial}{\partial v}\frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'.$$

We shall show that these conditions are sufficient.

When equations (35) are satisfied, equation (31) is true and there exists a surface Σ, defined by (32). Since the parametric curves on Σ are the characteristic lines, we have †

$$(36)\qquad \frac{\bar{D}}{E}=\frac{\bar{D}'}{G},\qquad \bar{D}'=0,$$

and consequently the second of equations (35) is reducible to ‡

$$(37)\qquad \frac{\partial}{\partial u}\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}_1=\frac{\partial}{\partial v}\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}_1,$$

where the Christoffel symbols are formed with respect to the linear element of Σ. This is the condition that there exist a surface S corresponding to Σ with orthogonality of linear elements and such that its asymptotic lines are parametric. § If we take S for the middle surface of a congruence of Ribaucour whose director-surface is Σ, we have a congruence of the kind sought.

Since the developables of a congruence of Ribaucour correspond to the asymptotic lines on the director surface, the conditions (10) that the characteristic ruled surfaces be parametric are equivalent to equations (36). Hence we have

THEOREM 7. *The characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in the curves which correspond to the characteristic conjugate system on the director-surface.*

* E., p. 420.
† Loc. cit., p. 435.
‡ E., p. 201.
§ E., p. 380.

In order that these curves on the middle surface may be asymptotic lines, the condition (37) must be satisfied, and consequently we have

THEOREM 8. *Let Σ be a surface of positive curvature whose characteristic lines form a conjugate system with equal point invariants and let S be the unique surface corresponding to Σ with orthogonality of linear elements in such a way that its asymptotic lines correspond to the characteristic lines on Σ; then S is the middle surface and Σ the director-surface of a congruence of Ribaucour whose characteristic ruled surfaces cut S in its asymptotic lines; and these are the only congruences of this sort.*

§ 4. *The Case $E = G$.*

We shall establish the existence of such congruences, by showing that there exist upon the unit sphere systems of curves which are such that when they are parametric the conditions $E = G$ and (35) are satisfied. The latter conditions become in this case

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}' = \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}', \qquad \frac{\partial}{\partial u}\begin{Bmatrix}11\\2\end{Bmatrix}' = \frac{\partial}{\partial v}\begin{Bmatrix}22\\1\end{Bmatrix}'. \tag{38}$$

If we put

$$E = G = \lambda, \qquad F = \lambda\cos\omega, \tag{39}$$

the first of these conditions may be written

$$\frac{\partial}{\partial u}\left(\frac{\dfrac{\partial \log\lambda}{\partial v} - \cos\omega\dfrac{\partial \log\lambda}{\partial u}}{\sin\omega}\right) = \frac{\partial}{\partial v}\left(\frac{\dfrac{\partial \log\lambda}{\partial u} - \cos\omega\dfrac{\partial \log\lambda}{\partial v}}{\sin\omega}\right),$$

and the second is reducible by means of the first to

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right),$$

or in other form

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\tan\frac{\omega}{2} = 0.$$

The general solution of this equation may be given the form

$$\tan^2\frac{\omega}{2} = \frac{\psi(u-v)}{\phi(u+v)}, \tag{40}$$

where ϕ and ψ are arbitrary functions.

From the first of equations (38) it follows that there is a surface S_0 whose asymptotic lines have the given representation on the sphere. Since $E = G$, the equation of the lines of curvature on S_0 is $du^2 - dv^2 = 0$, so that if

$$u_1 = u + v, \qquad v_1 = u - v,$$

the curves $u_1 = \text{const.}$, $v_1 = \text{const.}$ are lines of curvature. From (40) it follows

that

$$\cos\omega = \frac{\phi(u+v)-\psi(u-v)}{\phi(u+v)+\psi(u-v)} = \frac{U(u_1)-V(v_1)}{U(u_1)+V(v_1)}, \tag{40'}$$

and consequently the linear element of the spherical representation is

$$d\sigma^2 = \frac{\lambda}{U+V}(Udu_1^2 + Vdv_1^2).$$

Moreover, the linear element of S_0 is of the form *

$$ds_0^2 = \mu\left(\frac{du_1^2}{U} + \frac{dv_1^2}{V}\right),$$

where μ is a determinate function. Hence S_0 is an isothermic surface whose lines of curvature are represented on the sphere by an isothermal system. Conversely, it can be shown that when a surface of this kind is referred to its asymptotic lines conditions (38) and (40) are satisfied. †

To this group of surfaces belong the quadrics, cyclides of Dupin, surfaces of revolution, minimal surfaces, certain surfaces with plane lines of curvature in both systems, ‡ and a group of systems recently discussed by A. E. Young. §

Suppose that S_0 is such a surface referred to its asymptotic lines, and that S_1 is the unique surface whose characteristic lines have the same spherical representation as S_0. In consequence of (39) these lines on S_1 form an isothermal-conjugate system, and since the second of equations (38) is satisfied, this conjugate system has equal point invariants. ‖ Hence there exists a surface S, corresponding to S_1 with orthogonality of linear elements, whose asymptotic lines correspond to the characteristic lines on S_1. From Theorem 7 it follows that the congruence of Ribaucour for which S is the middle surface and S_1 the director surface is such that the characteristic ruled surfaces cut S in its asymptotic lines, and thus we have a congruence of the kind sought.

It is evident that S_0 is an associate surface of S_1 and consequently determines a surface S' corresponding to S_1 with orthogonality of linear elements. When S_0 is referred to its asymptotic lines, the parametric curves on S' form a conjugate system. In view of Theorem 7 we have that the characteristic ruled surfaces of the congruence of Ribaucour, whose director-surface is S_1 and middle surface S_1', cut S' in a conjugate system.

In general, when the characteristic ruled surfaces of a congruence of Ribaucour

* E., p. 192.

† Cf. A. E. YOUNG, *On a certain class of isothermic surfaces*, Transactions of the American Mathematical Society, vol. 10 (1909), pp. 79–94.

‡ EISENHART, *Isothermal-conjugate systems of lines on surfaces*, American Journal of Mathematics, vol. 25 (1902), pp. 239–248.

§ l. c.

‖ E., p. 380.

cut the middle surface in a conjugate system, the spherical representation of these surfaces is that of the asymptotic lines on that associate surface of the director-surface which is determined by the middle surface. Hence we have

THEOREM 9. *A necessary and sufficient condition that the characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in a conjugate system is that the spherical representation of these surfaces satisfy the conditions*

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\}' = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\}',$$

$$\frac{\partial^2 \log \frac{E}{G}}{\partial u \partial v} + \frac{\partial}{\partial u}\left(\frac{G}{E}\left\{\begin{matrix}11\\2\end{matrix}\right\}'\right) - \frac{\partial}{\partial v}\left(\frac{E}{G}\left\{\begin{matrix}22\\1\end{matrix}\right\}'\right) = 0. \tag{41}$$

The knowledge of one such system of curves on the sphere leads to the determination of another. This results from

THEOREM 10. *If the characteristic lines on a surface correspond to the asymptotic lines on an associate surface, the characteristic lines on the latter correspond to the asymptotic lines on the former.*

For, let S_1 and S_0 be two associate surfaces with the characteristic lines and asymptotic lines respectively parametric; then

$$\frac{D_1}{E_1} = \frac{D_1''}{G_1}, \qquad D_1' = 0, \qquad D_0 = D_0'' = 0, \qquad D_0' = -\mu D_1'' = \sigma D_1, \tag{42}$$

where μ and σ are two functions such that the coördinates of the two surfaces are in the relations

$$\frac{\partial x_0}{\partial u} = -\mu \frac{\partial x_1}{\partial v}, \qquad \frac{\partial x_0}{\partial v} = \sigma \frac{\partial x_1}{\partial u},$$

and similarly for the y's and z's.* From these we have

$$E_0 = \mu^2 G_1, \qquad G_0 = \sigma^2 E_1, \tag{43}$$

the functions E_1, G_1 in (42) and (43) being the coefficients of the linear element of S_1, and E_0, G_0 the corresponding functions for S_0. The differential equation of the characteristic lines on S_0 is reducible to†

$$E_0 du^2 + G_0 dv^2 = 0,$$

which by means of (42) and (43) is equivalent to

$$D_1 du^2 + D_1'' dv^2 = 0;$$

consequently the theorem is proved.

* E., pp. 378, 380.

† E., p. 131.

Since asymptotic lines and characteristic lines are real only on surfaces of negative and positive curvature respectively, it follows that if a system satisfying (41) is real, the similar system obtained by means of Theorem 10 is imaginary, and *vice-versa*.

§ 5. *Isothermic Systems of Ruled Surfaces. The Pseudofocal Surfaces.*

Since the quadratic form

$$\Phi \equiv Adu^2 + 2Bdudv + Cdv^2$$

is definite for a congruence of the elliptic type, there exist double systems of ruled surfaces which are such that, when a system of this kind is parametric, the coefficients of the form Φ satisfy the conditions

$$A = C, \qquad B = 0. \tag{44}$$

The determination of these double systems of ruled surfaces is the same analytical problem as that of isothermic orthogonal systems of curves on a surface.* For this reason we say that ruled surfaces satisfying the conditions (44) form an *isothermic system*.

On the assumption that such a system is parametric, we have from (6)

$$Ef' + Gf - F(e+g) = 0, \qquad Eg - Ge + F(f'-f) = 0. \tag{45}$$

Furthermore, if we take the middle surface of the congruence for the surface of reference, we have the condition (13). Combining the latter and (45), we find that

$$\frac{e}{F} = -\frac{f}{E} = \frac{f'}{G} = -\frac{g}{F} = \sigma, \tag{46}$$

where σ denotes the factor of proportionality.

When the values (46) are substituted in the first two of equations (22), we obtain

$$\gamma = -\frac{\partial \sigma}{\partial v} - \sigma S, \qquad \gamma_1 = \frac{\partial \sigma}{\partial u} + \sigma T, \tag{47}$$

where we have put

$$S = \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}', \qquad T = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}'. \tag{48}$$

And the third of equations (22) is reducible to

$$\frac{\partial^2 \sigma}{\partial u^2} + \frac{\partial^2 \sigma}{\partial v^2} + T\frac{\partial \sigma}{\partial u} + S\frac{\partial \sigma}{\partial v} + \left(\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + E + G \right) \sigma = 0. \tag{49}$$

* Cf., E., p. 93.

Moreover, from (21) we derive the equations of the middle surface in the form

$$\begin{aligned}\frac{\partial x}{\partial u} &= \sigma \frac{\partial X}{\partial v} - \left(\frac{\partial \sigma}{\partial v} + \sigma S\right) X,\\ \frac{\partial x}{\partial v} &= -\sigma \frac{\partial X}{\partial u} + \left(\frac{\partial \sigma}{\partial u} + \sigma T\right) X.\end{aligned} \tag{50}$$

Conversely, when the parametric lines on the sphere are any whatever, and σ is any solution of equation (49), equations (50) define the coördinates of the middle surface of a congruence referred to an isothermic system of ruled surfaces.*

We shall consider now the two surfaces S_1 and S_2, whose coördinates are given by

$$\begin{aligned} x_1 &= x - \sigma X, \qquad y_1 = y - \sigma Y, \qquad z_1 = z - \sigma Z,\\ x_2 &= x + \sigma X, \qquad y_2 = y + \sigma Y, \qquad z_2 = z + \sigma Z.\end{aligned} \tag{51}$$

In the first place we remark that S_1 and S_2, thus defined, are the same surfaces for all isothermic systems. In fact, if the values (46) be substituted in the second of (24), we find the following relation between σ and the abscissa of an imaginary focal point

$$\sigma = i\rho. \tag{52}$$

Because of this result we call S_1 and S_2 the *pseudofocal surfaces* of the congruence.

§ 6. *Conjugate-Potential Surfaces. Congruences of Lilienthal.*

Of particular interest is the case when the tangent planes at corresponding points of S_1 and S_2 are parallel. In order to investigate this case, we differentiate equations (51), with the result

$$\begin{aligned}\frac{\partial x_1}{\partial u} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) - \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma S\right) X,\\ \frac{\partial x_1}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} + \sigma T\right) X,\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{\partial x_2}{\partial u} &= \sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right) X,\\ \frac{\partial x_2}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma T\right) X.\end{aligned} \tag{54}$$

If X_1, Y_1, Z_1; X_2, Y_2, Z_2 denote the direction-cosines of the normals to S_1

*Cf. SANNIA, *Nuova esposizione della geometria infinitesimale delle congruenze rettilinee*, Annali di matematica, ser. 3, vol. 15 (1908), pp. 39, 40.

and S_2, we find from (53) and (54)

$$X_1 = \frac{\sigma^2}{\sqrt{E_1G_1 - F_1^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(2\frac{\partial \log\sigma}{\partial u} + S + T\right)\right.$$
$$\left. + \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\frac{\partial \log\sigma}{\partial v} + S - T\right)\right\},$$

(55)

$$X_2 = \frac{\sigma^2}{\sqrt{E_2G_2 - F_2^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(-2\frac{\partial \log\sigma}{\partial u} + S - T\right)\right.$$
$$\left. - \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\frac{\partial \log\sigma}{\partial v} + S + T\right)\right\},$$

where E_1, F_1, G_1; E_2, F_2, G_2 are the first fundamental coefficients of S_1 and S_2. From these expressions it follows that the necessary and sufficient condition that the tangent planes to S_1 and S_2 be parallel is that

$$E_1G_1 - F_1^2 = E_2G_2 - F_2^2 \tag{56}$$

and

$$\frac{\partial \log\sigma}{\partial u} = -\tfrac{1}{2}T, \qquad \frac{\partial \log\sigma}{\partial v} = -\tfrac{1}{2}S. \tag{57}$$

When the values from (57) are substituted in (53) and (54), it is found that

$$\frac{\partial x_1}{\partial u} = \frac{\partial x_2}{\partial v} = -\sigma\left[\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \tfrac{1}{2}(S - T)X\right],$$
$$\frac{\partial x_1}{\partial v} = -\frac{\partial x_2}{\partial u} = -\sigma\left[\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) - \tfrac{1}{2}(S + T)X\right]. \tag{58}$$

From these equations it follows that $x_1 + ix_2$, $y_1 + iy_2$, $z_1 + iz_2$ are analytic functions of $u + iv$. Hence we say that S_1 and S_2 are *conjugate-potential surfaces*, and the congruence consists of the joins of corresponding points of the pair.

Conversely, given any three analytic functions, consider the congruence of linesjoining corresponding points of the surfaces S_1 and S_2 so determined. The direction-cosines of the lines are of the form

$$X = \frac{x_2 - x_1}{2\sigma}, \qquad Y = \frac{y_2 - y_1}{2\sigma}, \qquad Z = \frac{z_2 - z_1}{2\sigma},$$

where 2σ denotes the distance between the points on S_1 and S_2. From these we obtain

$$\frac{\partial X}{\partial u} = -\frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial v} + \frac{\partial x_1}{\partial u} + 2\frac{\partial \log\sigma}{\partial u}X\right),$$
$$\frac{\partial X}{\partial v} = \frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial u} - \frac{\partial x_1}{\partial v} - 2\frac{\partial \log\sigma}{\partial v}X\right). \tag{59}$$

If x, y, z denote the coördinates of the mid-point of the join of corresponding

points of S_1 and S_2, equations (59) may be given the form

$$(60)\quad \frac{\partial X}{\partial u} = -\frac{1}{\sigma}\left(\frac{\partial x}{\partial v} + 2\frac{\partial \log \sigma}{\partial u} X\right), \qquad \frac{\partial X}{\partial v} = \frac{1}{\sigma}\left(\frac{\partial x}{\partial u} - 2\frac{\partial \log \sigma}{\partial v} X\right).$$

From these equations follow equations (46). Hence the surface S is the middle surface of the congruence, and the parametric ruled surfaces form an isothermic system. Congruences of this kind were considered by Lilienthal,* and so we shall refer to them as *congruences of Lilienthal.*

The preceding results may be stated thus:

THEOREM 11. *A necessary and sufficient condition that the pseudofocal surfaces of a congruence correspond with parallelism of tangent planes is that the congruence be of the Lilienthal type; in this case the pseudofocal surfaces are conjugate-potential.*

From (57) it follows that a necessary condition that the parametric lines on the sphere represent an isothermic system of ruled surfaces of a congruence of Lilienthal is

$$(61)\quad \frac{\partial}{\partial v} T = \frac{\partial}{\partial u} S.$$

Furthermore, the function σ given by (57) must satisfy equation (49). This gives the further condition

$$(62)\quad 2\left[\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + 2(E+G)\right] = T^2 + S^2.$$

Conversely, from the general theory it follows that when conditions (61) and (62) are satisfied, there exists a congruence of Lilienthal which can be found by quadratures.

In consequence of the identity †

$$\frac{\partial}{\partial v}(\{{}^{11}_{1}\}' + \{{}^{12}_{2}\}') = \frac{\partial}{\partial u}(\{{}^{22}_{2}\}' + \{{}^{12}_{1}\}'),$$

equation (61) is equivalent to

$$\frac{\partial}{\partial v}(\{{}^{12}_{2}\}' - \{{}^{22}_{1}\}') = \frac{\partial}{\partial u}(\{{}^{12}_{1}\}' - \{{}^{11}_{2}\}').$$

But this is the condition ‡ that the parametric lines on the sphere represent a real isothermal-conjugate system on a surface Σ whose coördinates ξ, η, ζ are given by the equations

$$(63)\quad \frac{\partial \xi}{\partial u} = \frac{t}{H^2}\left(-G\frac{\partial X}{\partial u} + F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial \xi}{\partial v} = \frac{t}{H^2}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right),$$

* *Untersuchungen zur allgemeinen Theorie der krummen Oberflächen und geradlinigen Strahlensysteme* (Bonn, 1886), p. 80.

† E., p. 153.

‡ E., p. 202.

where t is given by

$$\frac{\partial \log t}{\partial u} = \{\begin{smallmatrix}12\\2\end{smallmatrix}\}' - \{\begin{smallmatrix}22\\1\end{smallmatrix}\}', \qquad \frac{\partial \log t}{\partial v} = \{\begin{smallmatrix}12\\1\end{smallmatrix}\}' - \{\begin{smallmatrix}11\\2\end{smallmatrix}\}'.$$

From (50) and (63) we find that

$$dx\,d\xi + dy\,d\eta + dz\,d\zeta = 0.$$

Hence:

THEOREM 12. *A congruence of Lilienthal is a congruence of Ribaucour.*

If we put

$$u = \bar{u} + \bar{v}, \qquad v = i(\bar{v} - \bar{u}),$$

the ruled surfaces $\bar{u} = \text{const.}$, $\bar{v} = \text{const.}$ are the developables. If $\bar{E}$, $\bar{F}$, $\bar{G}$ denote the fundamental coefficients of the sphere in this case, we find that the conditions (61) and (62) may be given the form

$$\frac{\partial}{\partial \bar{u}} \{\begin{smallmatrix}\bar{1}\bar{2}\\1\end{smallmatrix}\}' = \frac{\partial}{\partial \bar{v}} \{\begin{smallmatrix}\bar{1}\bar{2}\\2\end{smallmatrix}\}' = \{\begin{smallmatrix}\bar{1}\bar{2}\\1\end{smallmatrix}\}' \{\begin{smallmatrix}\bar{1}\bar{2}\\2\end{smallmatrix}\}' - \bar{F}, \tag{64}$$

where the symbols $\{\begin{smallmatrix}\bar{r}\bar{s}\\t\end{smallmatrix}\}$ are formed with respect to $\bar{E}d\bar{u}^2 + 2\bar{F}d\bar{u}\,d\bar{v} + \bar{G}d\bar{v}^2$.

When a real system of lines on the sphere satisfies (64), these lines represent the developables of a congruence whose focal surfaces are curves;* moreover, the semi-focal distance is given by quadratures.

Since the tangent planes to two conjugate-potential surfaces S_1 and S_2 at corresponding points are parallel, it is readily shown that the second fundamental coefficients of these surfaces satisfy the relations

$$D_1 = -D_1'' = D_2', \qquad D_1' = -D_2 = D_2'',$$

and consequently

$$D_1 D_2'' + D_1' D_2 - 2D_1' D_2' = 0.$$

Hence we have

THEOREM 13. *Conjugate-potential surfaces are associate to each other.*

§ 7. *Congruences which consist of the Lines joining Corresponding Points on Associate Surfaces.*

We have seen that the abscissæ of the focal points are $i\sigma$, $-i\sigma$; hence, the above result follows also from the following known theorem †:

THEOREM 14. *In order that two surfaces S_1 and S_2 corresponding with parallelism of tangent planes be associate surfaces, it is necessary and sufficient that for the congruence formed by the joins of corresponding points M_1 and M_2 of these surfaces, the focal points and the points M_1 and M_2 form a harmonic range.*

* E., p. 412.

† E., p. 425.

We inquire whether a pair of associate surfaces is connected with every congruence after the manner described in the preceding theorem.

We assume that a given congruence possesses this property and that the two associate surfaces meet a line of the congruence in points whose distances from the middle point of the line are denoted by t_1 and t_2. By Theorem 14 we have $t_1 t_2 = \rho^2$, and by (52)

$$t_1 t_2 = -\sigma^2. \tag{65}$$

We must express the condition that the surfaces Σ_1 and Σ_2, defined by

$$\xi_1 = x + t_1 X, \qquad \eta_1 = y + t_1 Y, \qquad \zeta_1 = z + t_1 Z,$$
$$\xi_2 = x + t_2 X, \qquad \eta_2 = y + t_2 Y, \qquad \zeta_2 = z + t_2 Z,$$

correspond with parallelism of tangent planes.

By means of (50) we have

$$\begin{aligned} \frac{\partial \xi_1}{\partial u} &= t_1 \frac{\partial X}{\partial u} + \sigma \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) X, \\ \frac{\partial \xi_1}{\partial v} &= -\sigma \frac{\partial X}{\partial u} + t_1 \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) X, \end{aligned} \tag{66}$$

and similar expressions in η_1 and ζ_1. From these we find

$$\begin{aligned} \sqrt{E_1 G_1 - F_1^2}\, X_1 = (t_1^2 + \sigma^2) HX &+ \left[t_1 \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) \right. \\ &\left. + \sigma \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) \right] \left(Z \frac{\partial Y}{\partial u} - Y \frac{\partial Z}{\partial u} \right) \\ &+ \left[\sigma \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) - t_1 \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) \right] \left(Z \frac{\partial Y}{\partial v} - Y \frac{\partial Z}{\partial v} \right), \end{aligned}$$

where E_1, F_1, G_1 are the first fundamental coefficients of Σ_1 and X_1, Y_1, Z_1 are the direction courses of the normal to Σ_1.

The expressions for X_2, Y_2, Z_2 are similar to the above. The necessary and sufficient conditions that these respective quantities be equal are reducible to

$$t_1 \sqrt{E_2 G_2 - F_2^2} = -t_2 \sqrt{E_1 G_1 - F_1^2}, \tag{67}$$

$$\frac{\partial \theta}{\partial v} + 4\sigma \frac{\partial \sigma}{\partial u} + 2\sigma^2 T + S\theta = 0, \qquad \frac{\partial \theta}{\partial u} - 4\sigma \frac{\partial \sigma}{\partial v} - 2\sigma^2 S + T\theta = 0 \tag{68}$$

where, for the sake of brevity, we have put

$$\theta = t_1 \sigma - \frac{\sigma^3}{t_1}. \tag{69}$$

By means of (49) the condition of integrability of (68) is reducible to

$$\frac{1}{4}\left(\frac{\partial S}{\partial u}-\frac{\partial T}{\partial v}\right)\theta+\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)$$
$$-\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0. \tag{70}$$

Hence in order that two associate surfaces Σ_1 and Σ_2 exist it is necessary that the function θ given by (70) satisfy equations (68). Moreover, it is readily shown that these conditions are sufficient.

When in particular $\theta=0$, equations (68) reduce to (57) and then (70) is satisfied. This is the case of congruences of Lilienthal.

It may be shown that equation (61) is equivalent to the first of equations (64) and consequently is the condition that the congruence be of the Ribaucour type. From this and (70) it results that for the congruences of Ribaucour, for which σ is a solution of (49) and

$$\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)$$
$$-\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0, \tag{71}$$

the function θ involves a parameter, and consequently there is an infinity of pairs of associate surfaces such as Σ_1 and Σ_2. Moreover, their determination requires quadratures only.

Since congruences of Lilienthal are of the Ribaucour type, the above result is applicable to these congruences; for equation (71) is satisfied by the function σ given by (57). Now equations (68) reduce to

$$\frac{\partial \theta}{\partial v}-2\frac{\partial \log \sigma}{\partial v}\cdot\theta=0, \qquad \frac{\partial \theta}{\partial u}-2\frac{\partial \log \sigma}{\partial u}\theta=0,$$

of which the general solution is

$$\theta=c\sigma^2,$$

c being an arbitrary constant. From this result, (69), and (65), we have

$$t_1=a\sigma, \qquad t_2=-\frac{\sigma}{a},$$

where a is an arbitrary constant. Hence we have

THEOREM 15. *With a congruence of Lilienthal there are associated an infinity of pairs of associate surfaces; two of these are conjugate-potential surfaces which cut a line of the congruence at points distant σ and $-\sigma$ from the middle point; and corresponding points of any other pair are at distances $a\sigma$, $-\sigma/a$, where a is a constant.*

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